HOPF-ORE EXTENSIONS AND POINTED HOPF ALGEBRAS OF RANK ONE

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Abstract. In this paper, we study pointed rank one Hopf algebras and Hopf-Ore extensions of group algebras, over an arbitrary field k. It is proved that the rank of a Hopf-Ore extension of a group algebra is one or two or infinite. It is also shown that an arbitrary (finite or infinite dimensional) pointed Hopf algebra of rank one is isomorphic to a quotient of a Hopf-Ore extension of its coradical, a group algebra. We classify the finite dimensional simple modules and describe a family of indecomposable modules over a Hopf-Ore extension $H = kG(\chi, a, \delta)$ and its quotient H' of rank one, where $\chi(a) \neq 1$, G is an abelian group and k is an algebraically closed field. The decomposition of the tensor products of two finite dimensional simple modules into a direct sum of indecomposable modules is given too. We also determine all simple objects and a family of indecomposable projective objects in the categories of all weight modules over H and H'.

Introduction

A ring R is said to be an invariant basis number (IBN) ring if for every free left R-module M, any two bases of M have the same cardinal, which is defined to be the rank of the free module M in this case. For a Hopf algebra H over a field k, let $H_0 \subseteq H_1 \subseteq H_2 \subseteq \cdots$ denote the coradical filtration of H. Krop and Radford defined the rank as a measure of complexity for Hopf algebras in [3], which equals the rank of H_1/H_0 as a free H_0-module. The definition of the rank of a Hopf algebra H depends on the IBN property of H_0.

Let H be a Hopf algebra over a field k. Assume that the coradical H_0 is a Hopf subalgebra of H, and H is generated, as an algebra, by H_1. Let k be the trivial right H_0-module given by the counit ε, H_1 a left H_0-module by multiplication. Then the rank of H is defined to be $\dim_k(k \otimes_{H_0} H_1) - 1$ (see [3]). Krop and Radford classified all finite dimensional pointed Hopf algebras of rank one over k, where k is an algebraically closed field of characteristic 0, and gave a presentation of these Hopf algebras by generators and relations in [3]. Scherotzke studied such Hopf algebras for the case of $\mathrm{char}(k) = p > 0$ in [7].

In this paper, we study Hopf-Ore extensions of group algebras and classify the (infinite or finite dimensional) pointed Hopf algebras H of rank one over an arbitrary field k. We also investigate the representations of these Hopf algebras. The paper is organized as follows. In Section 1, we present some basic definitions and facts about Hopf algebras and Hopf-Ore extensions. In Section 2, for any group G, we compute the rank of Hopf-Ore extension $kG[x; \tau, \delta] = kG(\chi, a, \delta)$ of kG, where τ is an automorphism of kG, δ is a τ-derivation of kG, χ is a k-valued character for G and $a \in Z(G)$, the center of G. We

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show the following results: if $\mathrm{char}(k) = 0$ and $\chi(a)$ is an n-th primitive root of unity for some $n \geq 2$, then the rank of $kG(\chi, a, \delta)$ is two; if $\mathrm{char}(k) = p > 0$ and $\chi(a)$ is an N-th primitive root of unity for some $N \geq 1$, then the rank of $kG(\chi, a, \delta)$ is infinite; otherwise, the rank of $kG(\chi, a, \delta)$ is one. In Section 3, we classify the (infinite or finite dimensional) pointed Hopf algebras H of rank one over k. We prove that such a Hopf algebra H is a quotient of Hopf-Ore extension of its coradical $H_0 = kG(H)$, where $G(H)$ denotes the group of the group-like elements of H. Hence we obtain a presentation of these Hopf algebras by generators and relations. In Section 4, we study the representation theory of the Hopf algebra $H = kG(\chi, a, \delta)$, the Hopf-Ore extension of kG, and its quotient Hopf algebra $H' = H/I$ of rank one, where $\chi(a) \neq 1$, G is an abelian group and k is an algebraically closed field. We classify all finite dimensional simple modules and a family of indecomposable modules over H and H'. We describe the decompositions of the tensor products of two finite dimensional simple modules into a direct sum of indecomposable modules. We also determine the simple objects and a class of indecomposable projective objects in the categories of all weight modules over H and H'.

1. **Preliminaries**

Throughout, we work over a field k. Unless otherwise stated, all vector spaces, tensor products and linear maps are taken over k, and dim means $\dim_k$. Our reference for basic concepts and notations about Hopf algebras is [4]. In particular, for a Hopf algebra, we will use ε, Δ and S to denote the counit, comultiplication and antipode, respectively. We use Sweedler's notations for the comultiplication and coaction. Let $\mathbb{Z}$ be the set of all integers, and $\mathbb{N}$ the set of all nonnegative integers. Let $k^\times = k \setminus \{0\}$. For a group G, let $\hat{G}$ denote the set of characters of G over k, and let $Z(G)$ denote the center of G.

Let C be a coalgebra. The coradical C_0 of C is defined to be the sum of all simple subcoalgebras of C. C is called pointed if every simple subcoalgebra of C is one dimensional. The coradical filtration $\{C_n\}_{n\in\mathbb{N}}$ of C is defined inductively as follows

$$C_n := \Delta^{-1}(C \otimes C_{n-1} + C_0 \otimes C),\ n \geq 1.$$

Then we have the following lemma (See [4, Theorem 5.2.2]).

Lemma 1.1. $\{C_n\}_{n\geq 0}$ *are a family of subcoalgebras of C satisfying*

(1) $C_n \subseteq C_{n+1}$ *for all* $n \geq 0$ *and* $C = \bigcup_{n\geq 0} C_n$,

(2) $\Delta C_n \subseteq \sum_{i=0}^{n} C_i \otimes C_{n-i}$ *for all* $n \geq 0$.

Any set of subspaces of C satisfying conditions (1) and (2) in Lemma 1.1 is called a coalgebra filtration of C.

Now let us recall the concept of the rank of a Hopf algebra defined in [3].

Let H be a bialgebra. Assume that $V_{(0)} \subseteq V_{(1)} \subseteq V_{(2)} \subseteq \cdots$ is a series of subspaces of H satisfying $\Delta V_{(n)} \subseteq \sum_{i=0}^{n} V_{(i)} \otimes V_{(n-i)}$ for all $n \geq 0$. Then $V_{(n)}$ is a subcoalgebra of H for any $n \geq 0$. Let $V = \bigcup_{n=0}^{\infty} V_{(n)}$. Then V is a subcoalgebra of H, and $V_{(0)} \subseteq V_{(1)} \subseteq V_{(2)} \subseteq \cdots$ is a coalgebra filtration of V. Suppose further that $V_{(0)}$ is also a subalgebra of H and has an

antipode. Then $K = V_{(0)}$ is a Hopf algebra. Assume that $V_{(n)}$ is a K-submodule of H under left multiplication for all $n \geq 0$.

Recall that H is a left H-comodule under the comultiplication. In this case, all subcoalgebras of H are left subcomodules of H. Let $V_{(-1)} = 0$ and $Q_{(n)} = V_{(n)}/V_{(n-1)}$ for all $n \geq 0$. Since $\Delta V_{(n)} \subseteq K \otimes V_{(n)} + V_{(n)} \otimes V_{(n-1)}$, it follows that $Q_{(n)}$ is a left K-comodule for all $n \geq 0$. Thus with the quotient left K-module structure and the left K-comodule structure above, $Q_{(n)}$ is a left K-Hopf module. Therefore, $Q_{(n)}$ is a free left K-module with a K-basis being any linear basis of $Q^{\natural}_{(n)} = \{z \in Q_{(n)} | \rho_n(z) = 1 \otimes z\}$ by the Fundamental Theorem of Hopf modules [4, Theorem 1.9.4], where ρ_n is the comodule structure map of $Q_{(n)}$. It follows that $V_{(n)}$ is a free left K-module for any $n \geq 0$, and consequently V is a free left K-module.

Let M be a free left K-module, k the trivial right K-module. Then any K-basis of M has $\mathrm{Dim}_k(k \otimes_K M)$ elements, which is observed in [3]. This means that K is an IBN ring.

Theorem 1.2. *A bialgebra is an IBN ring. In particular, a Hopf algebra is an IBN ring.*

Proof. Let B be a bialgebra with the counit $\varepsilon : B \to k$. Since ε is a nonzero epimorphism and k is an IBN ring (In fact, any commutative ring with identity is an IBN ring [1, Corollary 2.12]), it follows from [1, Proposition 2.11] that B is also an IBN ring. □

Let H be a Hopf algebra, $H_0 \subseteq H_1 \subseteq H_2 \subseteq \cdots$ the coradical filtration of H. Assume that H_0 is a Hopf subalgebra of H. Then each H_i is a free H_0-module.

Definition 1.3. (See [3]) Let H be a Hopf algebra, $H_0 \subseteq H_1 \subseteq H_2 \subseteq \cdots$ the coradical filtration of H. Assume that H_0 is a Hopf subalgebra. If H is generated by H_1 as an algebra and $\mathrm{Dim}_k(k \otimes_{H_0} H_1) = n + 1$, then we say that H is a Hopf algebra of rank n.

A coalgebra C is graded if there exist subspaces $\{C_{(n)}\}_{n\geq 0}$ of C such that $C = \bigoplus_{n\geq 0} C_{(n)}$ and $\Delta(C_{(n)}) \subseteq \sum_{i=0}^{n} C_{(i)} \otimes C_{(n-i)}$ for all $n \geq 0$ and $\varepsilon(C_{(n)}) = 0$ for all $n > 0$. If this is the case, then $C \otimes C$ is also a graded coalgebra with grading $(C \otimes C)_{(n)} = \sum_{i=0}^{n} C_{(i)} \otimes C_{(n-i)}$. Moreover, the comultiplication $\Delta : C \to C \otimes C$ of C is a graded map.

Recall that a group-like element in a coalgebra C is an element $g \in C$ with $\Delta(g) = g \otimes g$ and $\varepsilon(g) = 1$. The set of group-like elements in C is denoted by $G(C)$. For $g, h \in G(C)$, $c \in C$ is called (g, h)-primitive if $\Delta(c) = c \otimes g + h \otimes c$. The set of all (g, h)-primitives is denoted by $P_{g,h}(C)$. Let H be a Hopf algebra. Then $G(H)$ is a group. For g, $h \in G(H)$, if $c \in P_{g,h}(H)$, then $g^{-1}c \in P_{1,g^{-1}h}(H)$ and $ch^{-1} \in P_{gh^{-1},1}(H)$.

Now we introduce the notion of Hopf-Ore extension.

Let A be a k-algebra. Let τ be an algebra endomorphism of A and δ a τ-derivation of A. The Ore extension $A[y; \tau, \delta]$ of the algebra A is an algebra generated by the variable y and the algebra A with the relation $ya = \tau(a)y + \delta(a)$ for all $a \in A$. If $\{a_i \mid i \in I\}$ is a k-basis of A, $\{a_i y^j \mid i \in I, j \in \mathbb{N}\}$ is a k-basis of $A[y; \tau, \delta]$ (see [5]).

Definition 1.4. ([5, Definition 1.0]) Assume that A and $A[y; \tau, \delta]$ are Hopf algebras. The Hopf algebra $A[y; \tau, \delta]$ is called a Hopf-Ore extension of A if $\Delta(y) = y \otimes r_1 + r_2 \otimes y$ for some r_1, $r_2 \in A$ and A is a Hopf subalgebra of $A[y; \tau, \delta]$.

Remark 1.5. In Definition 1.4, one can easily check that both r_1 and r_2 are group-like elements. Replacing the generating element y by yr_2^{-1}, we can assume that the element y satisfies the relation

$$\Delta(y) = y \otimes r + 1 \otimes y$$

for some group-like element $r \in A$. Here we make some modification for $\Delta(y)$, which is assumed to be $\Delta(y) = y \otimes 1 + r \otimes y$ in [5]. Hence there are also some corresponding changes in the following Theorem 1.6 and related results from the original ones in [5].

If $A[y;\tau,\delta]$ is a Hopf-Ore extension of A, then $\varepsilon(y) = 0$ and $S(y) = -yr^{-1}$, where S is the antipode of $A[y;\tau,\delta]$.

Theorem 1.6. *(*[5, Theorem 1.3]*) Assume that A and $A[y;\tau,\delta]$ are Hopf algebras. Then $A[y;\tau,\delta]$ is a Hopf-Ore extension of A if and only if the following conditions are satisfied:*

(a) *there is a character $\chi : A \to k$ such that $\tau(a) = \sum a_1\chi(a_2)$ for all $a \in A$;*

(b) *$\sum a_1\chi(a_2) = \sum \chi(a_1)ad_r(a_2)$ for all $a \in A$, where $ad_r(a) = rar^{-1}$;*

(c) *$\Delta\delta(a) = \sum \delta(a_1) \otimes ra_2 + \sum a_1 \otimes \delta(a_2)$ for all $a \in A$.*

Let $\delta \in \mathrm{End}_k(A)$. δ is called an r-coderivation if $\Delta\delta(a) = \sum \delta(a_1) \otimes ra_2 + \sum a_1 \otimes \delta(a_2)$, $a \in A$. δ is said to be a $\langle\chi, r\rangle$-derivation if δ is a τ-derivation and an r-coderivation, where χ is a character of A determined by τ as in Theorem 1.6(a).

Notation 1.7. Denote the Hopf-Ore extension $A[y;\tau,\delta]$ by $A(\chi, r, \delta)$, where $\chi : A \to k$ is a character, r is a group-like element of A as in Theorem 1.6, and δ is a $\langle\chi, r\rangle$-derivation.

Let $A = kG$ be a group algebra, χ a character of kG. A linear map $\alpha : kG \to k$ is called a 1-cocycle of G with respect to χ if $\alpha(gh) = \alpha(g) + \chi(g)\alpha(h)$ for all $g, h \in G$. Let $Z^1_\chi(G)$ denote the set of all 1-cocycles of G with respect to χ.

Theorem 1.8. *(*[5, Proposition 2.2]*) Every Hopf-Ore extension of $A = kG$ is of the form $A(\chi, r, \delta)$, where χ is a group character, $r \in Z(G)$, and $\delta(a) = \sum a_1(1 - r)\alpha(a_2)$ for some $\alpha \in Z^1_\chi(G)$.*

Let $0 \neq q \in k$. For any integer $n > 0$, set $(n)_q = 1 + q + \cdots + q^{n-1}$. Observe that $(n)_q = n$ when $q = 1$, and

$$(n)_q = \frac{q^n - 1}{q - 1}$$

when $q \neq 1$. Define the q-factorial of n by $(0)!_q = 1$ and $(n)!_q = (n)_q(n-1)_q \cdots (1)_q$ for $n > 0$. Note that $(n)!_q = n!$ when $q = 1$, and

$$(n)!_q = \frac{(q^n - 1)(q^{n-1} - 1)\cdots(q - 1)}{(q - 1)^n}$$

when $n > 0$ and $q \neq 1$. The q-binomial coefficients $\begin{pmatrix} n \\ m \end{pmatrix}_q$ is defined inductively as follows for $0 \leq m \leq n$:

$$\begin{pmatrix} n \\ 0 \end{pmatrix}_q = 1 = \begin{pmatrix} n \\ n \end{pmatrix}_q \qquad \text{for } n \geq 0,$$

$$\begin{pmatrix} n \\ m \end{pmatrix}_q = q^m \begin{pmatrix} n-1 \\ m \end{pmatrix}_q + \begin{pmatrix} n-1 \\ m-1 \end{pmatrix}_q \qquad \text{for } 0 < m < n.$$

It is well-known that $\begin{pmatrix} n \\ m \end{pmatrix}_q$ is a polynomial in q with integer coefficients and with value at $q = 1$ equal to the usual binomial coefficients $\begin{pmatrix} n \\ m \end{pmatrix}$, and that

$$\begin{pmatrix} n \\ m \end{pmatrix}_q = \frac{(n)!_q}{(m)!_q(n-m)!_q}$$

when $(n-1)!_q \neq 0$ and $0 < m < n$ (see [2, Page 74]).

Lemma 1.9. *(*[6, Corollary 2]*) Let* $0 \neq q \in k$ *and* $n > 1$*. Then* $\begin{pmatrix} n \\ m \end{pmatrix}_q = 0$ *for all* $1 \leq m < n$ *if and only if*

(a) *char*$(k) = 0$ *and* q *is a primitive* n*-th root of unity, or*

(b) *char*$(k) = p > 0$ *and* q *is a primitive* N*-th root of unity, where* $n = Np^r$ *with* $p \nmid N$ *(*$N \geq 1$, $r \geq 0$*).*

2. The ranks of Hopf-Ore extensions of a group algebra

In this section, assume that $H = kG[x; \tau, \delta]$ is a Hopf-Ore extension of kG, where G is a group. By Theorem 1.8, $H = kG(\chi, a, \delta)$, where $\chi \in \hat{G}$, $a \in Z(G)$ and $\delta(g) = g(1-a)\alpha(g)$, $g \in G$, for some $\alpha \in Z^1_\chi(G)$.

We have that $\alpha(gh) = \alpha(g) + \chi(g)\alpha(h)$ for all $g, h \in G$. It follows that

$$\alpha(g^i) = \alpha(g)(1 + \chi(g) + \cdots + \chi(g)^{i-1}),\ g \in G,\ i \geq 1,$$

and $\alpha(1) = 0$. Hence $\alpha(g^{-1}) = -\chi(g^{-1})\alpha(g)$, $g \in G$.

By the definition of Hopf-Ore extension, we have that

$$g^{-1}xg = \chi(g)x + \alpha(g)(1-a),\ \Delta(x) = x \otimes a + 1 \otimes x,\ g \in G.$$

Now we can simplify the presentation of $H = kG(\chi, a, \delta)$ by changing the generator x, depending on the values of $\chi(a)$ and $\alpha(a)$.

Case 1: $\chi(a) \neq 1$. In this case, set $x' = x - \frac{\alpha(a)}{1-\chi(a)}(1-a)$. Then for any $g \in G$, we have

$$\begin{aligned} g^{-1}x'g &= g^{-1}xg - \tfrac{\alpha(a)}{1-\chi(a)}(1-a) \\ &= \chi(g)x + \alpha(g)(1-a) - \tfrac{\alpha(a)}{1-\chi(a)}(1-a) \\ &= \chi(g)x' + (\tfrac{\chi(g)\alpha(a)}{1-\chi(a)} + \alpha(g) - \tfrac{\alpha(a)}{1-\chi(a)})(1-a) \\ &= \chi(g)x' + \tfrac{1}{1-\chi(a)}(\chi(g)\alpha(a) + \alpha(g) - \alpha(g)\chi(a) - \alpha(a))(1-a) \\ &= \chi(g)x' + \tfrac{1}{1-\chi(a)}(\alpha(ga) - \alpha(ag))(1-a) \\ &= \chi(g)x'. \end{aligned}$$

H is generated, as an algebra, by G and x' . Thus, replacing the generator x by x', one can assume that $g^{-1}xg = \chi(g)x$, $g \in G$, and $\Delta(x) = x \otimes a + 1 \otimes x$ in this case.

Case 2: $\chi(a) = 1$ and $\alpha(a) = 0$. In this case, $a^{-1}xa = x$.

Note that if $|a| = n < \infty$ with $\chi(a) = 1$, $\mathrm{char}(k) = 0$, or $\mathrm{char}(k) = p > 0$ and $p \nmid n$, then we have

$$0 = \alpha(a^n) = \alpha(a)(1 + \chi(a) + \cdots + \chi(a)^{n-1}) = n\alpha(a),$$

which implies that $\alpha(a) = 0$.

Case 3: $\chi(a) = 1$ and $\alpha(a) \neq 0$. In this case, we have $a^{-1}xa = x + \alpha(a)(1 - a)$. Set $x' = \alpha(a)^{-1}x$. Then $a^{-1}x'a = x' + (1 - a)$ and $\Delta(x') = x' \otimes a + 1 \otimes x'$. Moreover, H is generated, as an algebra, by G and x'. Hence by replacing x with x', one can assume that $a^{-1}xa = x + (1 - a)$ and $\Delta(x) = x \otimes a + 1 \otimes x$ in this case.

Lemma 2.1. *Assume that B is a bialgebra. Let $a \in G(B)$ and $x \in B$ with $\Delta(x) = x \otimes a + 1 \otimes x$. Let $q, \beta \in k$ with $q \neq 0$. Then we have the followings:*

(a) ([3, Eq. (1)]) *If $xa = qax$, then*

$$\Delta(x^n) = \sum_{l=0}^{n} \binom{n}{l}_q x^{n-l} \otimes a^{n-l}x^l, \quad n > 0.$$

(b) *If $xa = qax + \beta a(1 - a)$, then for any $n > 0$ we have*

$$\Delta(x^n) = \sum_{l=0}^{n} \binom{n}{l}_q x^{n-l} \otimes a^{n-l}x^l + P_n, \quad P_n \in \sum_{i+j \leq n-1} (B(i) \otimes B(j)),$$

where $B(i) = \mathrm{span}\{a^l x^i \mid l \geq 0\}$ for $i \geq 0$.

(c) *(*[7, Corollary 4.10]*) Assume that char$(k) = p > 0$. If $xa = ax + a(1 - a)$, then*

$$\Delta(x^p) = x^p \otimes a^p + (p - 1)!x \otimes a^p + x \otimes a + 1 \otimes x^p,$$

(d) *Assume that char$(k) = p > 0$. If $xa = ax + a(1 - a)$, then for all $r \geq 1$,*

$$\Delta(x^{p^r} - x^{p^{r-1}}) = (x^{p^r} - x^{p^{r-1}}) \otimes a^{p^r} + 1 \otimes (x^{p^r} - x^{p^{r-1}}).$$

Proof. Part (b) can be shown by induction on n using (a). For Part (d), by Part (c), we have

$$\begin{aligned} \Delta(x^p - x) &= x^p \otimes a^p + (p - 1)!x \otimes a^p + x \otimes a + 1 \otimes x^p - x \otimes a - 1 \otimes x \\ &= (x^p - x) \otimes a^p + 1 \otimes (x^p - x). \end{aligned}$$

Note that $xa^p = a^p x$, and so $((x^p - x) \otimes a^p)(1 \otimes (x^p - x)) = (1 \otimes (x^p - x))((x^p - x) \otimes a^p)$. Thus, when $r \geq 2$, we have

$$\begin{aligned} \Delta(x^{p^r} - x^{p^{r-1}}) &= \Delta[(x^p - x)^{p^{r-1}}] = [\Delta(x^p - x)]^{p^{r-1}} \\ &= [(x^p - x) \otimes a^p + 1 \otimes (x^p - x)]^{p^{r-1}} \\ &= [(x^p - x) \otimes a^p]^{p^{r-1}} + [1 \otimes (x^p - x)]^{p^{r-1}} \\ &= (x^{p^r} - x^{p^{r-1}}) \otimes a^{p^r} + 1 \otimes (x^{p^r} - x^{p^{r-1}}). \end{aligned}$$

□

Note that H has a k-basis $\{gx^i \mid g \in G, i \geq 0\}$. Let $H(n) = \mathrm{span}\{gx^n \mid g \in G\}$ and $H[n] = \mathrm{span}\{gx^i \mid g \in G, 0 \leq i \leq n\}$, $n \geq 0$. Then $H[0] = H(0) = H_0 = kG$ and

$H[n] = \bigoplus_{i=0}^{n} H(i)$ for $n > 0$. By Lemma 2.1, $H[0] \subseteq H[1] \subseteq H[2] \subseteq \cdots$ is a coalgebra filtration of H.

In the rest of this section, write $q = \chi(a)$ for simplicity.

Proposition 2.2. *In **Cases 1** and **2**, $H = \bigoplus_{n=0}^{\infty} H(n)$ is a graded coalgebra.*

Proof. Since $a^{-1}xa = qx$ in **Cases 1** and **2**, it follows from Lemma 2.1(a) that $\Delta(H(n)) \subseteq \bigoplus_{i+j=n} (H(i) \otimes H(j))$ for all $n \in \mathbb{N}$. Obviously, $\varepsilon(H(n)) = 0$ for all $n \geq 0$. This completes the proof. □

Lemma 2.3. *Assume that $C = \bigoplus_{i=0}^{\infty} C_{(i)}$ is a graded coalgebra. Let $g, h \in G(C)$ and let $y = \sum_{i=0}^{n} y_i \in C$ with $y_i \in C_{(i)}$. Then y is a (g, h)-primitive element if and only if each y_i is a (g, h)-primitive element.*

Proof. Since $C = \bigoplus_{i=0}^{\infty} C_{(i)}$ is a graded coalgebra, the comultiplication Δ preserves the grading. Hence

$$\Delta y_i \in (C \otimes C)_{(i)} := \bigoplus_{j=0}^{i} C_{(j)} \otimes C_{(i-j)}, \quad 0 \leq i \leq n.$$

By [4, Lemma 5.3.4], $kG(C) \subseteq C_0 \subseteq C_{(0)}$, and hence $y_i \otimes g + h \otimes y_i \in (C \otimes C)_{(i)}$ for all $0 \leq i \leq n$. Now we have $\Delta(y) = \sum_{i=0}^{n} \Delta(y_i)$ and $y \otimes g + h \otimes y = \sum_{i=0}^{n} (y_i \otimes g + h \otimes y_i)$. It follows that $\Delta(y) = y \otimes g + h \otimes y$ if and only if $\Delta(y_i) = y_i \otimes g + h \otimes y_i$ for all $0 \leq i \leq n$. □

Lemma 2.4. ([4, Theorem 5.4.1]) *Assume that C is a pointed coalgebra. Let $C_0 = kG(C) \subset C_1 \subset C_2 \subset \cdots$ be the coradical filtration of C. Then*

(a) *$C_1 = kG(C) \oplus (\oplus_{g,h \in G(C)} P'_{g,h}(C))$, where $P'_{g,h}(C)$ is a subspace of $P_{g,h}(C)$ such that $P_{g,h}(C) = k(g-h) \oplus P'_{g,h}(C)$,*

(b) *for any $n \geq 1$ and $c \in C_n$, $c = \sum_{g,h \in G} c_{g,h}$, where $\Delta c_{g,h} = c_{g,h} \otimes g + h \otimes c_{g,h} + w$ for some $w \in C_{n-1} \otimes C_{n-1}$.*

We now analyze the term H_1 in the coradical filtration of H.

Theorem 2.5. (a) *Assume $char(k) = 0$. If q is a primitive n-th root of unity for some $n \geq 2$, then the rank of H is 2 and $H_1 = H_0 \oplus H_0 x \oplus H_0 x^n$; otherwise, the rank of H is 1 and $H_1 = H_0 \oplus H_0 x$.*

(b) *Assume $char(k) = p > 0$. If q is a primitive N-th root of unity for some $N \geq 2$, then the rank of H is infinite and $H_1 = H_0 \oplus H_0 x \oplus (\oplus_{r \geq 0} H_0 x^{Np^r})$; if $q = 1$, then the rank of H is infinite and $H_1 = H_0 \oplus (\oplus_{r \geq 0} H_0 x^{p^r})$; otherwise, the rank of H is 1 and $H_1 = H_0 \oplus H_0 x$.*

Proof. By Lemma 2.4(a) we know that H_1 is spanned by all group-like elements and skew primitive elements. Hence we need to compute all (g, h)-primitive elements for all group-like elements g, h. Note that $y \in H$ is a (g, h)-primitive element if and only if $h^{-1}y$ is an

$(h^{-1}g, 1)$-primitive element. Hence we only need to describe the $(g, 1)$-primitive elements for any $g \in G$.

We first consider **Cases 1** and **2**. In these cases, we can assume that

$$a^{-1}xa = qx, \quad \Delta(x) = x \otimes a + 1 \otimes x.$$

Let $0 \neq h \in H$ with $\Delta(h) = h \otimes g + 1 \otimes h$ for some $g \in G$. By Proposition 2.2 and Lemma 2.3, we only need to consider the case that $h \in H(i) = H_0x^i$, $i \geq 0$. If $h \in H_0$, then $h = \beta(1 - g)$ for some $\beta \in k^{\times}$. If $h \in H_0x$, then $h = \sum_{b\in G} \beta_b bx, \beta_b \in k$. Comparing the both sides of the equation

$$\sum_{b\in G} \beta_b(bx \otimes ba + b \otimes bx) = (\sum_{b\in G} \beta_b bx) \otimes g + 1 \otimes (\sum_{b\in G} \beta_b bx),$$

we have that $\beta_b = 0$ if $b \neq 1$. Hence $h = \beta_1 x$, and so $g = a$. Thus

$$h = \beta_1 x, \quad \Delta(h) = h \otimes a + 1 \otimes h.$$

If $h \in H_0x^n$ for some $n \geq 2$, then $h = \sum_{b\in G} \beta_b bx^n, \beta_b \in k$. Hence we have

$$\sum_{b\in G} \beta_b(\sum_{l=0}^{n} \begin{pmatrix} n \\ l \end{pmatrix}_q bx^{n-l} \otimes ba^{n-l}x^l) = (\sum_{b\in G} \beta_b bx^n) \otimes g + 1 \otimes (\sum_{b\in G} \beta_b bx^n),$$

from which one knows that $\beta_b = 0$ if $b \neq 1$. Thus, $h = \beta_1 x^n$, $\begin{pmatrix} n \\ l \end{pmatrix}_q = 0$ for all $1 \leq l \leq n - 1$, and $g = a^n$. By Lemma 1.9, $\mathrm{char}(k) = 0$ and q is a primitive n-th root of unity, or $\mathrm{char}(k) = p > 0$ and q is a primitive N-th root of unity, where $n = Np^r$ with $p \nmid N$, $N \geq 1$ and $r \geq 0$. This proves the theorem for **Cases 1** and **2**.

Now we consider **Case 3**. In this case,

$$a^{-1}xa = x + (1 - a) \quad \Delta(x) = x \otimes a + 1 \otimes x.$$

Let $0 \neq h = \sum_{i=0}^{n} \sum_{b\in G} \beta(i, b)bx^i \in H$, where $\beta(i, b) \in k$, and $\beta(n, b) \neq 0$ for some $b \in G$. Assume $\Delta(h) = h \otimes g + 1 \otimes h$ for some $g \in G$. Then

$$\begin{aligned} &\sum_{i=0}^{n} \sum_{b\in G} \sum_{l=0}^{i} \beta(i, b) \begin{pmatrix} i \\ l \end{pmatrix} bx^{i-l} \otimes ba^{i-l}x^l + \sum_{i=0}^{n} \sum_{b\in G} \beta(i, b)(b \otimes b)P_i \\ = \ &\sum_{i=0}^{n} \sum_{b\in G} \beta(i, b)bx^i \otimes g + 1 \otimes \sum_{i=0}^{n} \sum_{b\in G} \beta(i, b)bx^i, \end{aligned}$$

where P_i's are the elements described as in Lemma 2.1(b). Note that $\{x^i \otimes x^j \mid i, j \geq 0\}$ is a basis of $H \otimes H$ as a free $H_0 \otimes H_0$-module, and $\{bx^i \otimes cx^j \mid i, j \geq 0,\ b, c \in G\}$ is a k-basis of $H \otimes H$. So we can compare the coefficients of the two sides of the above equation.

(1) If $n = 0$, then obviously $h = \beta(1 - g)$ for some $\beta \in k^{\times}$.

Now we consider the case of $n \geq 1$. By comparing the coefficients of $1 \otimes x^n$, one gets that $\beta(n, b) = 0$ for all $b \neq 1$. Hence $\beta(n, 1) \neq 0$. By comparing the coefficients of $x^n \otimes 1$, one gets that $g = a^n$.

(2) If $n = 1$, then $g = a$ and $h = h_0 + \beta x \in H[1]$ for some $h_0 \in H(0)$ and $\beta \in k^{\times}$. Since $H[1] = H(0) \oplus H(1)$ is a graded coalgebra, by Lemma 2.3, a similar argument as above shows that $h = \beta_0(1 - a) + \beta x$ for some $\beta_0 \in k$.

(3) If $n \geq 2$, by comparing the coefficients of $x^{n-l} \otimes x^l$ $(0 < l < n)$, one gets that $\beta(n,1)\begin{pmatrix} n \\ l \end{pmatrix} = 0$ for all $0 < l < n$. This forces $\mathrm{char}(k) = p > 0$ and $n = p^r$ for some $r \geq 1$ by Lemma 1.9. Set $n_1 = n$, $r_1 = r$, $h_1 = h$. Then $h_2 := h - \beta(p^{r_1},1)(x^{p^{r_1}} - x^{p^{r_1-1}})$ is an $(a^{p^{r_1}},1)$-primitive element by Lemma 2.1(d). If $0 \neq h_2 = \sum_{i=0}^{n_2}\sum_{b\in G}\gamma(i,b)bx^i$ such that $\gamma(n_2,b) \neq 0$ for some $b \in G$, and $n_2 \geq 2$, then a similar discussion as above shows that $n_2 = p^{r_2}$ for some $r_2 \geq 1$ and $a^{p^{r_1}} = a^{p^{r_2}}$. Continuing this procedure, we have that $h - \sum_{i=1}^{s}\beta_i(x^{p^{r_i}} - x^{p^{r_i-1}}) \in H[1]$, where $\beta_i \in k^\times$, $r_1 > r_2 > \cdots > r_s \geq 1$, $s \geq 1$, and $a^{p^{r_1}} = a^{p^{r_2}} = \cdots = a^{p^{r_s}}$. If $h - \sum_{i=1}^{s}\beta_i(x^{p^{r_i}} - x^{p^{r_i-1}}) = 0$, then $h = \sum_{i=1}^{s}\beta_i(x^{p^{r_i}} - x^{p^{r_i-1}})$. If $h - \sum_{i=1}^{s}\beta_i(x^{p^{r_i}} - x^{p^{r_i-1}}) \neq 0$, then it follows from (1) and (2) that either $h = \sum_{i=1}^{s}\beta_i(x^{p^{r_i}} - x^{p^{r_i-1}}) + \beta_0(1-g)$ with $\beta_0 \in k^\times$ and $g = a^{p^{r_1}} = a^{p^{r_2}} = \cdots = a^{p^{r_s}}$, or $h = \sum_{i=1}^{s}\beta_i(x^{p^{r_i}} - x^{p^{r_i-1}}) + \beta x + \beta_0(1-a)$ with $\beta \in k^\times, \beta_0 \in k$ and $a = a^{p^{r_1}} = a^{p^{r_2}} = \cdots = a^{p^{r_s}}$. This proves the theorem for **Case 3** by Lemma 2.1(d). □

By Theorem 2.5 and its proof, we have the following corollary.

Corollary 2.6. *Let $g \in G$ and $z \in H\backslash H_0$ with $\Delta(z) = z \otimes g + 1 \otimes z$.*

(a) *In the case of $\mathrm{char}(k) = 0$, we have*

(i) *if q is a primitive n-th root of unity with $n \geq 2$, then either $g = a$, $z = \gamma x + \beta(1-a)$, or $g = a^n$, $z = \gamma x^n + \beta(1-a^n)$, where $\gamma, \beta \in k$;*

(ii) *Otherwise, $g = a$, $z = \gamma x + \beta(1-a)$ for some $\gamma, \beta \in k$.*

(b) *In the case of $\mathrm{char}(k) = p > 0$, we have*

(i) *if q is a primitive N-th root of unity with $N \geq 2$, then either $g = a$, $z = \gamma x + \beta(1-a)$, or $g = a^{Np^r}$, $z = \gamma(1-a^{Np^r}) + \sum_{i=1}^{s}\beta_i x^{Np^{r_i}}$, where $\gamma, \beta, \beta_1 \cdots, \beta_s \in k$, $r, r_1, \cdots, r_s \in \mathbb{N}$ such that $a^{Np^{r_i}} = a^{Np^r}$;*

(ii) *if $q = 1$ and $a^{-1}xa = x$, then either $g = a$, $z = \gamma(1-a) + \sum_{i=1}^{s}\beta_i x^{p^{r_i}}$ for some $\gamma, \beta_1, \cdots, \beta_s \in k$ and $r_1, \cdots, r_s \in \mathbb{N}$ such that $a^{p^{r_i}} = a$, or $g = a^{p^r} \neq a$, $z = \gamma(1-a^{p^r}) + \sum_{i=1}^{s}\beta_i x^{p^{r_i}}$ for some $\gamma, \beta_1, \cdots, \beta_s \in k$ and $r, r_1, \cdots, r_s \in \mathbb{N}$ such that $r_i \geq 1$ and $a^{p^{r_i}} = a^{p^r}$;*

(iii) *if $q = 1$ and $a^{-1}xa = x + (1-a)$, then either $g = a$, $z = \gamma(1-a) + \beta x + \sum_{i=1}^{s}\beta_i(x^{p^{r_i}} - x^{p^{r_i-1}})$ for some $\gamma, \beta, \beta_1, \cdots, \beta_s \in k$ and $r_1, \cdots, r_s \in \mathbb{N}$ such that $r_i \geq 1$ and $a^{p^{r_i}} = a$, or $g = a^{p^r} \neq a$, $z = \gamma(1-a^{p^r}) + \sum_{i=1}^{s}\beta_i(x^{p^{r_i}} - x^{p^{r_i-1}})$ for some $\gamma, \beta_1, \cdots, \beta_s \in k$ and $r, r_1, \cdots, r_s \in \mathbb{N}$ such that $r_i \geq 1$ and $a^{p^{r_i}} = a^{p^r}$;*

(iv) *if q is not a root of unity, then $g = a$, $z = \gamma x + \beta(1-a)$ for some $\gamma, \beta \in k$.*

3. The classification of pointed Hopf algebras of rank one

In this section, we discuss the structures of pointed Hopf algebras of rank one over a field k. We will prove that such Hopf algebras are some Hopf quotients of Hopf-Ore extensions of their coradicals. [3] and [7] classified all finite dimensional pointed Hopf algebras of rank one over an algebraically closed field with $\mathrm{char}(k) = 0$ and $\mathrm{char}(k) = p > 0$, respectively. Actually, the classification results there still hold without the assumption that k is algebraically closed.

Proposition 3.1. *Let H be an arbitrary pointed Hopf algebra of rank one, $G = G(H)$. Then there are some $a \in G$ and $x \in H \setminus H_0$ such that $\Delta(x) = x \otimes a + 1 \otimes x$. Moreover, $H_1 = H_0 \oplus H_0 x$.*

Proof. Let $H_0 \subset H_1 \subset H_2 \subset \cdots$ be the coradical filtration of H. Then $Q_1 = H_1/H_0$ is left H_0-Hopf module, and hence it is a free H_0-module. Note that the left H_0-module action on H_1/H_0 is given by the multiplication in H, and the left H_0-comodule structure map $\rho : H_1/H_0 \to H_0 \otimes H_1/H_0$ satisfies $\rho\pi = (\mathrm{id} \otimes \pi)\Delta$, where $\pi : H_1 \to H_1/H_0$ is the natural epimorphism. Since H is a Hopf algebra of rank one, $Q_1^\sharp = \{z \in Q_1 \mid \rho(z) = 1 \otimes z\}$ is one dimensional over k. Hence any nonzero element of $Q_1^\sharp$ forms a basis of H_1/H_0 as a left H_0-module. Let $0 \neq z \in Q_1^\sharp$. By Lemma 2.4(a), there exists an element $y \in \oplus_{g,h \in G} P'_{g,h}(H)$ such that $z = \pi(y)$. Let $y = \sum_{g,h \in G} c_{g,h}$, where $c_{g,h} \in P'_{g,h}(H)$. Then

$$\Delta(y) = \sum_{g,h} (c_{g,h} \otimes g + h \otimes c_{g,h}),$$

so $1 \otimes \pi(y) = 1 \otimes z = \rho(z) = \rho\pi(y) = (\mathrm{id} \otimes \pi)\Delta(y) = \sum_{g,h} h \otimes \pi(c_{g,h}) = \sum_h (h \otimes \pi(\sum_g c_{g,h}))$, from which one knows that $\pi(\sum_g c_{g,h}) = 0$ if $h \neq 1$. It follows that $\sum_g c_{g,h} = 0$ if $h \neq 1$, and so $y = \sum_g c_{g,1}$. Thus, $z = \pi(y) = \sum_g \pi(c_{g,1}) \neq 0$, and hence there exists some $a \in G$ such that $c_{a,1} \neq 0$. Obviously, $0 \neq \pi(c_{a,1}) \in Q_1^\sharp$.

Let $x = c_{a,1}$. Then $x \in H_1 \setminus H_0$ and $\Delta(x) = x \otimes a + 1 \otimes x$. Moreover, $\pi(x)$ forms an H_0-basis of H_1/H_0, and so $H_1 = H_0 \oplus H_0 x$. □

In the rest of this section, assume that H is an arbitrary (infinite or finite dimensional) pointed Hopf algebra of rank one, and $G = G(H)$. In this case, $H_0 = kG$. By Proposition 3.1, there exist some $a \in G$ and $x \in H_1 \setminus H_0$ such that $H_1 = H_0 \oplus H_0 x$ and $\Delta(x) = x \otimes a + 1 \otimes x$. Moreover, H is generated by x and G as an algebra, $H_1 = H_0 \oplus H_0 x$ is a graded coalgebra, and $\{gx^i \mid g \in G, 0 \leq i \leq 1\}$ is a k-basis of H_1.

Proposition 3.2. (a) *If $g \in G$ and $z \in H \setminus H_0$ with $\Delta(z) = z \otimes g + 1 \otimes z$, then $g = a$ and $z = \gamma x + \beta(1 - a)$ for some $\gamma, \beta \in k$ with $\gamma \neq 0$.*

(b) *$a \in Z(G)$.*

(c) *If $a \neq 1$, then there exist a character $\chi \in \hat{G}$ and a 1-cocycle $\alpha \in Z^1_\chi(G)$ such that*

$$g^{-1}xg = \chi(g)x + \alpha(g)(1 - a), \quad g \in G.$$

(d) *If $a = 1$, then there exists a character $\chi \in \hat{G}$ such that $g^{-1}xg = \chi(g)x$, $g \in G$.*

Proof. (a) Let $g \in G$ and $z \in H \setminus H_0$ with $\Delta(z) = z \otimes g + 1 \otimes z$. Then $z \in H_1$. Since $H_1 = H_0 \oplus H_0 x$ is a graded coalgebra, by Lemma 2.3 we only need to consider the case of $z \in H_0 x$. Let $z \in H_0 x$. Then $z = \sum_{b \in G} \gamma_b bx$ for some $\gamma_b \in k$, and hence

$$\sum_{b \in G} \gamma_b (bx \otimes ba + b \otimes bx) = (\sum_{b \in G} \gamma_b bx) \otimes g + 1 \otimes (\sum_{b \in G} \gamma_b bx).$$

It follows that $g = a$ and $\gamma_b = 0$ for all $b \neq 1$. Thus $z = \gamma_1 x$ with $0 \neq \gamma_1 \in k$.

(b),(c),(d) For any $g \in G$, we have that $\Delta(g^{-1}xg) = g^{-1}xg \otimes g^{-1}ag + 1 \otimes g^{-1}xg$. By (a), one knows that $g^{-1}ag = a$ and $g^{-1}xg = \chi(g)x + \alpha(g)(1 - a)$ for some $\chi(g),\ \alpha(g) \in k$ with $\chi(g) \neq 0$. Hence $a \in Z(G)$.

We claim that $\chi \in \hat{G}$ and that $\alpha \in Z^1_\chi(G)$ if $a \neq 1$. In fact, let $g, h \in G$. Then, on one hand, $(gh)^{-1}x(gh) = \chi(gh)x + \alpha(gh)(1-a)$. On the other hand,

$$\begin{aligned}(gh)^{-1}x(gh) =& h^{-1}(g^{-1}xg)h \\ =& h^{-1}(\chi(g)x + \alpha(g)(1-a))h \\ =& \chi(g)(\chi(h)x + \alpha(h)(1-a)) + \alpha(g)(1-a) \\ =& \chi(g)\chi(h)x + (\chi(g)\alpha(h) + \alpha(g))(1-a).\end{aligned}$$

It follows that $\chi(gh) = \chi(g)\chi(h)$, and hence $\chi \in \hat{G}$. If $a \neq 1$, then we also have $\alpha(gh) = \chi(g)\alpha(h) + \alpha(g)$, which means that $\alpha \in Z^1_\chi(G)$. □

From Parts (c) and (d) of Proposition 3.2, we have

$$g^{-1}xg = \chi(g)x + \alpha(g)(1-a),\ g \in G$$

in H. By a discussion similar to the one about the Hopf-Ore extension in Section 2, one can clarify H in three types according to the values of $\chi(a)$ and $\alpha(a)$.

Type 1: $\chi(a) \neq 1$. In this case, $a \neq 1$, by setting $\gamma = \frac{\alpha(a)}{1-\chi(a)}$ and substituting x with $x - \gamma(1-a)$, one can assume that

$$g^{-1}xg = \chi(g)x,\ g \in G.$$

Type 2: $\chi(a) = 1$ and $\alpha(a) = 0$. In this case, we have

$$a^{-1}xa = x.$$

Type 3: $\chi(a) = 1$ and $\alpha(a) \neq 0$. In this case, we have $a^{-1}xa = x + \alpha(a)(1-a)$. By replacing x with $\alpha(a)^{-1}x$, one can assume that

$$a^{-1}xa = x + (1-a), \quad \Delta(x) = x \otimes a + 1 \otimes x.$$

Let $V_{(n)} = H_0 + H_0x + H_0x^2 + \cdots + H_0x^n$ for all $n \geq 0$. Since H is generated, as an algebra, by x and G, $V_{(0)} \subseteq V_{(1)} \subseteq V_{(2)} \subseteq \cdots$ is a coalgebra filtration of H. Now by an discussion similar to the one in [3, p.216], one can determine a basis of H as a free left H_0-module under multiplication. Set $V_{(-1)} = 0$, $Q_{(n)} = V_{(n)}/V_{(n-1)}$, and let $\pi_{(n)} : V_{(n)} \to Q_{(n)}$ be the canonical projection for all $n \geq 0$. Then $(Q_{(n)}, \rho_{(n)})$ is a left H_0-Hopf module for any $n \geq 0$ as described in Section 1, where the left H_0-action is induced by the left multiplication in H, and the left H_0-comodule structure map $\rho_{(n)}$ is determined by $\rho_{(n)}(\pi_{(n)}(v)) = (\mathrm{id} \otimes \pi_{(n)})\Delta(v)$ for all $v \in V_{(n)}$ and $n \geq 0$. Hence $Q_{(n)} = 0$ or $Q_{(n)}$ is a free left H_0-module with a basis being any linear basis of $Q^\natural_{(n)} = \{z \in Q_{(n)} \mid \rho_{(n)}(z) = 1 \otimes z\}$ by the fundamental theorem of Hopf modules [4, Theorem 1.9.4]. Consequently, $V_{(n)}$ is a free left H_0-module for any $n \geq 0$, and so is H. Since $Q_{(n)}$ is generated by $\pi_{(n)}(x^n)$ as a left H_0-module and $\rho_{(n)}(\pi_{(n)}(x^n)) = 1 \otimes \pi_{(n)}(x^n)$, it follows that $Q^\natural_{(n)}$ is spanned by $\pi_{(n)}(x^n)$. In particular, $Q_{(n)} = 0$ or $Q_{(n)}$ is a free left H_0-module with a basis $\{\pi_{(n)}(x^n)\}$.

Proposition 3.3. *H is a free left H_0-module under left multiplication with a basis $\{1, x, x^2, \cdots\}$ or $\{1, x, x^2, \cdots, x^{n-1}\}$ for some $n \geq 2$, where x is the $(a,1)$-primitive element in H as given above. Furthermore, in case that $\{1, x, x^2, \cdots, x^{n-1}\}$ is an H_0-basis of the free left H_0-module H, where $n \geq 2$, we have the following statements:*

(a) *If $char(k) = 0$, then $\chi(a)$ is a primitive n-th root of unity, and by a suitable substituting of x, we have that $g^{-1}xg = \chi(g)x$ for all $g \in G$, and $x^n = \beta(1-a^n)$ for some $\beta \in k$.*

(b) *If* $char(k) = p > 0$, *then either* $p \nmid n$ *and* $\chi(a)$ *is a primitive* n*-th root of unity, or* $n = p$ *and* $\chi(a) = 1$. *Moreover, we have*

(i) *if* $p \nmid n$, *then by a suitable substituting of* x, *we have that* $g^{-1}xg = \chi(g)x$ *for all* $g \in G$, *and* $x^n = \beta(1 - a^n)$ *for some* $\beta \in k$;

(ii) *if* $n = p$ *and* $a^{-1}xa = x$, *then* $x^p = \beta x + \gamma(1 - a^p)$ *for some* $\beta, \gamma \in k$. *Moreover,* $a^p = a$ *if* $\beta \neq 0$;

(iii) *if* $n = p$ *and* $a^{-1}xa = x + \alpha(a)(1 - a)$ *with* $\alpha(a) \neq 0$ *and* $a \neq 1$, *then by a suitable substituting of* x, *we have* $x^p = \beta x + \gamma(1 - a^p)$ *for some* $\beta \in k$. *Moreover,* $a^p = a$ *if* $\beta \neq 1$.

Proof. By the discussion before, an argument similar to the proof of [3, Lemma 1(a)] shows that that H is a free left H_0-module under the left multiplication with a basis $\{1, x, x^2, \cdots\}$ or $\{1, x, x^2, \cdots, x^{n-1}\}$ for some $n \geq 2$.

Now assume that $\{1, x, x^2, \cdots, x^{n-1}\}$ is an H_0-basis of the free left H_0-module H for some $n \geq 2$, and let $q = \chi(a)$. Then $H = H_0 + H_0 x + H_0 x^2 + \cdots + H_0 x^{n-1}$, and hence

$$x^n = b_0 + b_1 x + \cdots + b_{n-1} x^{n-1} = \sum_{l=0}^{n-1} b_l x^l$$

for some $b_0, b_1, \cdots, b_{n-1} \in H_0$. Note that $H \otimes H$ is a free left $H_0 \otimes H_0$-module with an $H_0 \otimes H_0$-basis $\{x^i \otimes x^j \mid 0 \leq i, j \leq n-1\}$. Let W be the $H_0 \otimes H_0$-submodule of $H \otimes H$ generated by $\{x^i \otimes x^j | i + j \leq n-1\}$, and let U be the $H_0 \otimes H_0$-submodule of $H \otimes H$ generated by $\{x^i \otimes x^j | i, j \leq n-1, i + j = n\}$. Then obviously $W \cap U = 0$. From Lemma 2.1, we have $\Delta(\sum_{l=0}^{n-1} b_l x^l) = \sum_{l=0}^{n-1} \Delta(b_l)\Delta(x^l) \in W$ and

$$\begin{aligned}
\Delta(x^n) &= \sum_{l=0}^{n} \binom{n}{l}_q x^l \otimes a^l x^{n-l} + P \\
&= \sum_{l=1}^{n-1} \binom{n}{l}_q x^l \otimes a^l x^{n-l} + 1 \otimes x^n + x^n \otimes a^n + P \\
&= \sum_{l=1}^{n-1} \binom{n}{l}_q x^l \otimes a^l x^{n-l} + 1 \otimes (\sum_{l=0}^{n-1} b_l x^l) + (\sum_{l=0}^{n-1} b_l x^l) \otimes a^n + P,
\end{aligned}$$

where $P \in W$. Since $1 \otimes (\sum_{l=0}^{n-1} b_l x^l) + (\sum_{l=0}^{n-1} b_l x^l) \otimes a^n \in W$ and $\sum_{l=1}^{n-1} \binom{n}{l}_q x^l \otimes a^l x^{n-l} \in U$, it follow from $\Delta(x^n) = \Delta(\sum_{l=0}^{n-1} b_l x^l) \in W$ that $\sum_{l=1}^{n-1} \binom{n}{l}_q x^l \otimes a^l x^{n-l} = 0$, which implies that $\binom{n}{l}_q = 0$ for all $1 \leq l \leq n-1$.

(a) Assume $\mathrm{char}(k) = 0$. Then by Lemma 1.9, q is a primitive n-th root of unity, and hence $q = \chi(a) \neq 1$. Thus, by a suitable substituting of x as stated in **Type 1**, we have $g^{-1}xg = \chi(g)x$ for all $g \in G$. In particular, $a^{-1}xa = qx$. If $x^n = 0$, then it is done. Now assume $x^n \neq 0$. Then $x^n = \sum_{i=0}^{m} b_i x^i$ for some $b_0, b_1, \cdots, b_m \in H_0$ with $b_m \neq 0$, where $0 \leq m \leq n-1$. By Lemma 2.1(a) and Lemma 1.9, we have $\Delta(x^n) = x^n \otimes a^n + 1 \otimes x^n = \sum_{i=0}^{m} b_i x^i \otimes a^n + 1 \otimes \sum_{i=0}^{m} b_i x^i = \sum_{i=0}^{m}((b_i \otimes a^n)(x^i \otimes 1) + (1 \otimes b_i)(1 \otimes x^i))$ and $\Delta(\sum_{i=0}^{m} b_i x^i) =$

$\sum_{i=0}^{m}\sum_{j=0}^{i}\begin{pmatrix} i \\ j \end{pmatrix}_q \Delta(b_i)(1\otimes a^j)(x^j\otimes x^{i-j})$. If $m \geq 2$, then by comparing the coefficients of the basis element $x^j\otimes x^{m-j}$ in $\Delta(x^n)$ and $\Delta(\sum_{i=0}^{m} b_i x^i)$, one gets that $\begin{pmatrix} m \\ j \end{pmatrix}_q \Delta(b_m)(1\otimes a^j) = 0$ for all $1 \leq j \leq m-1$. However, $\Delta(b_m)(1\otimes a^j) \neq 0$, and so $\begin{pmatrix} m \\ j \end{pmatrix}_q = 0$ for all $1 \leq j \leq m-1$. This is impossible by Lemma 1.9. Hence $m \leq 1$. If $m = 1$, then by comparing the coefficients of the basis elements $1\otimes x$ and $x\otimes 1$ in $\Delta(x^n)$ and $\Delta(b_0 + b_1 x)$, one gets that $1\otimes b_1 = \Delta(b_1)$ and $b_1\otimes a^n = \Delta(b_1)(1\otimes a)$, which implies that $b_1 \in k$ and $b_1 a^n = b_1 a$. It follows that $a^n = a$, and so $a^{n-1} = 1$ since $b_1 \neq 0$. This is also impossible since $n \geq 2$ and $q = \chi(a)$ is a primitive n-th root of unity. Thus, $x^n = b_0 \in H_0$ and $\Delta(x^n) = x^n\otimes a^n + 1\otimes x^n$. It follows that $x^n = \beta(1-a^n)$ for some $\beta \in k$.

(b) Assume char$(k) = p > 0$. Then by Lemma 1.9, q is a primitive N-th root of unity, where $n = Np^r$ with $p \nmid N$, $N \geq 1$ and $r \geq 0$. Since H is a Hopf algebra of rank one, it follows from the proof of Theorem 2.5 that either $p \nmid n$ (i.e. $N = n \geq 2$, $r = 0$) and q is a primitive n-th root of unity, or $n = p$ (i.e. $N = 1$, $r = 1$) and $q = 1$. If q is a primitive n-th root of unity (i.e. $p \nmid n$), then Part (b)(i) follows from an argument similar to Part (a). In case that $q = 1$ (i.e. $n = p$) and $a^{-1}xa = x$, an argument similar to Part (a) shows that $x^p = b_0 + b_1 x$ for some $b_0, b_1 \in H_0$. Then Part (b)(ii) follows by comparing the terms in $\Delta(x^p)$ and $\Delta(b_0 + b_1 x)$. If $q = 1$ and $a^{-1}xa = x + \alpha(a)(1-a)$ with $\alpha(a) \neq 0$ and $a \neq 1$, then by a suitable substituting of x, we can assume that $a^{-1}xa = x + (1-a)$. In this case, using Lemma 2.1(b) and (c), an argument similar to Part (a) shows that $x^p = b_0 + b_1 x$ for some $b_0, b_1 \in H_0$. Now by Lemma 2.1(c), we have

$$\begin{aligned}\Delta(x^p) &= x^p\otimes a^p - x\otimes a^p + x\otimes a + 1\otimes x^p \\ &= (b_0 + b_1 x)\otimes a^p - x\otimes a^p + x\otimes a + 1\otimes(b_0 + b_1 x) \\ &= (b_0\otimes a^p + 1\otimes b_0) + (b_1\otimes a^p - 1\otimes a^p + 1\otimes a)(x\otimes 1) + (1\otimes b_1)(1\otimes x),\end{aligned}$$

and $\Delta(b_0 + b_1 x) = \Delta(b_0) + \Delta(b_1)(x\otimes a + 1\otimes x) = \Delta(b_0) + \Delta(b_1)(1\otimes a)(x\otimes 1) + \Delta(b_1)(1\otimes x)$. It follows that $\Delta(b_0) = b_0\otimes a^p + 1\otimes b_0$, $\Delta(b_1)(1\otimes a) = b_1\otimes a^p - 1\otimes a^p + 1\otimes a$ and $\Delta(b_1) = 1\otimes b_1$. From $\Delta(b_0) = b_0\otimes a^p + 1\otimes b_0$, one gets that $b_0 = \gamma(1-a^p)$ for some $\gamma \in k$. From $\Delta(b_1) = 1\otimes b_1$, one gets $b_1 \in k$. Then from $\Delta(b_1)(1\otimes a) = b_1\otimes a^p - 1\otimes a^p + 1\otimes a$, one gets that $(b_1 - 1)(a - a^p) = 0$, which implies that $a^p = a$ if $b_1 \neq 1$. This shows Part (b)(iii). Thus, we complete the proof of Part (b). □

We have already described the algebra structure of H in Proposition 3.2 and Proposition 3.3. Now we give another description of H using the Hopf-Ore extension as follows. For any $y \in H$, let $\langle y\rangle$ denote the ideal of H generated by y.

Theorem 3.4. *H is isomorphic to a quotient of Hopf-Ore extension of its coradical $H_0 = kG$, where $G = G(H)$. Precisely, there exist a $\chi \in \hat{G}$, $a \in Z(G)$ and $\alpha \in Z^1_\chi(G)$ such that $H \cong kG(\chi, a, \delta)/I$ as Hopf algebras, where the derivation δ is defined by $\delta(g) = g(1-a)\alpha(a)$, $g \in G$, I is a Hopf ideal of $kG(\chi, a, \delta)$ defined as follows:*

(a) *In the case of char$(k) = 0$,*
 (i) *if $\chi(a)$ is a primitive n-th root of unity for some $n \geq 2$, then $I = \langle x^n - \beta(1-a^n)\rangle$ for some $\beta \in k$;*

(ii) *if $\chi(a)$ is not a primitive n-th root of unity for any $n \geq 2$, then $I = 0$.*

(b) *In the case of $char(k) = p > 0$,*

(i) *if $\chi(a)$ is a primitive n-th root of unity for some $n \geq 2$, then $I = \langle x^n - \beta(1 - a^n)\rangle$ for some $\beta \in k$;*

(ii) *if $\chi(a) = 1$ and $a^{-1}xa = x$, then $I = \langle x^p - \beta x - \gamma(1 - a^p)\rangle$ for some $\beta, \gamma \in k$. Moreover, $a^p = a$ if $\beta \neq 0$;*

(iii) *if $\chi(a) = 1$ and $a^{-1}xa = x + \alpha(a)(1 - a)$ with $\alpha(a) \neq 0$ and $a \neq 1$, then by a suitable substituting of x, $I = \langle x^p - \beta x - \gamma(1 - a^p)\rangle$ for some $\beta, \gamma \in k$. Moreover, $a^p = a$ if $\beta \neq 1$;*

(iv) *if $\chi(a)$ is not a primitive n-th root of unity for any $n \geq 1$, $I = 0$.*

Proof. Let $G = G(H)$. Then $H_0 = kG$. By Proposition 3.1, there exist some $a \in G$ and $x \in H_1 \setminus H_0$ such that $H_1 = H_0 \oplus H_0x$ and $\Delta(x) = x \otimes a + 1 \otimes x$. Moreover, H is generated by x and G as an algebra. By Proposition 3.2, $a \in Z(G)$, and there exist a character $\chi \in \hat{G}$ and a 1-cocycle $\alpha \in Z^1_\chi(G)$ such that

$$g^{-1}xg = \chi(g)x + \alpha(g)(1 - a), \quad g \in G.$$

Moreover, we can make the following assumption: $\alpha(g) = 0$ for all $g \in G$ when $\chi(a) \neq 1$; $\alpha(a) = 1$ when $\chi(a) = 1$ and $\alpha(a) \neq 0$. Then one can form a Hopf-Ore extension $kG(\chi, a, \delta)$ of kG, where $\delta \in \mathrm{End}(kG)$ is defined by $\delta(g) = g(1 - a)\alpha(g)$ for all $g \in G$. By the definition of Hopf-Ore extensions, there exists an algebra epimorphism $F : kG(\chi, a, \delta) \to H$ such that $F(x) = x$ and $F(g) = g$ for all $g \in G$. It is easy to see that F is also a coalgebra morphism. Consequently, F is a Hopf algebra epimorphism. Note that $kG(\chi, a, \delta)$ is a free left kG-module with the kG-basis $\{x^i | i \geq 0\}$. Obviously, F is a kG-module homomorphism. Let $I = \mathrm{Ker}(F)$. Then I is a Hopf ideal of $kG(\chi, a, \delta)$, and $H \cong kG(\chi, a, \delta)/I$ as Hopf algebras.

(a)(i) Assume that $\chi(a)$ is a primitive n-th root of unity for some $n \geq 2$. Then $\Delta(x^n) = x^n \otimes a^n + 1 \otimes x^n$ in H, and hence $x^n \in H_1 = H_0 \oplus H_0x$. It follows from Proposition 3.3 that H is a free left H_0-module with a basis $\{1, x, x^2, \cdots, x^{m-1}\}$ for some $2 \leq m \leq n$. By Proposition 3.3(a), $\chi(a)$ is a primitive m-th root of unity. Hence $m = n$ and H has a left H_0-module basis $\{1, x, x^2, \cdots, x^{n-1}\}$. Moreover, $x^n = \beta(1 - a^n)$ for some $\beta \in k$, in H. It follows that $I = \langle x^n - \beta(1 - a^n)\rangle$.

(a)(ii) If $\chi(a)$ is not a primitive n-th root of unity for any $n \geq 2$, then H is a free left H_0-module with the basis $\{1, x, x^2, \cdots\}$ by Proposition 3.3. Obviously, F is an isomorphism in this case. Hence $I = 0$ and $H \cong kG(\chi, a, \delta)$ as Hopf algebras.

(b) It is similar to Part (a) using Proposition 3.3.

□

Let $|\chi|$ denote the order of the character χ.

Proposition 3.5. *With the notations in Theorem 3.4, assume that $\chi(a)$ is a primitive n-th root of unity for some $n \geq 2$. Then we have*

(a) $n \leq |\chi|$.

(b) *If* $|\chi| < \infty$, *then* $n \| |\chi|$.

(c) *If* $n < |\chi|$, *then* $I = \langle x^n - \beta(1-a^n)\rangle = \langle x^n, \beta(1-a^n)\rangle$.

Proof. Parts (a) and (b) are obvious. For Part (c), we may assume that $g^{-1}xg = \chi(g)x$ for all $g \in G$ as stated in the proof of Theorem 3.4. Since $n < |\chi|$, there exists some $g \in G$ such that $\chi^n(g) \neq 1$. Then $x^n \in I$ since $x^n - \beta(1-a^n) \in I$ and $\chi^n(g)x^n - \beta(1-a^n) = g^{-1}(x^n - \beta(1-a^n))g \in I$. Hence $I = \langle x^n - \beta(1-a^n)\rangle = \langle x^n, \beta(1-a^n)\rangle$. □

Corollary 3.6. *With the notations in Theorem 3.4, assume that* $\chi(a)$ *is a primitive n-th root of unity for some* $n \geq 2$. *Then we have*

(a) *If* $\beta = 0$ *or* $a^n = 1$, *then* $I = \langle x^n \rangle$.

(b) *If* $\beta \neq 0$ *and* $a^n \neq 1$ *and* $n < |\chi|$, *then* $I = \langle x^n, 1 - a^n\rangle$.

(c) *If* $\beta \neq 0$ *and* $a^n \neq 1$ *and* $n = |\chi|$, *then* $I = \langle x^n - \beta(1-a^n)\rangle$.

Proof. It follows from Theorem 3.4 and Proposition 3.5. □

Remark 3.7. Let $G = G(H)$ and $H_0 = kG$. When $\chi(a)$ is a primitive n-th root of unity for some $n \geq 2$, $H = H_0 + H_0x + H_0x^2 + \cdots + H_0x^{n-1}$. When char$k = p > 0$ and $\chi(a) = 1$, $H = H_0 + H_0x + H_0x^2 + \cdots + H_0x^{p-1}$. In these cases, if H is infinite dimensional, then G is an infinite group, and H has the similar structures and properties with the finite dimensional pointed Hopf algebras of rank one classified in [3, 7] except that G is infinite.

4. **Representations**

In this section, we study the representation theory of Hopf-Ore extensions of group algebras and pointed Hopf algebras of rank one. By the discussion in Section 3, any pointed Hopf algebra of rank one is isomorphic to a quotient of the Hopf-Ore extension of its coradical.

In what follows, let $H = kG(\chi^{-1}, a, \delta)$ be a Hopf-Ore extension of a group algebra kG. Then $H' = kG(\chi^{-1}, a, \delta)/I$ is a Hopf algebra of rank one, where I is a Hopf ideal of H as described in Theorem 3.4. Moreover, every pointed Hopf algebra of rank one has this form. Let $\pi : H \to H'$ be the canonical epimorphism.

Let ${}_H\mathcal{M}$ (resp. ${}_{H'}\mathcal{M}$) denote the category of the left H-modules (resp. H'-modules), which is also denoted by $\mathcal{M}$ (resp. $\mathcal{M}'$) for simplicity. Then both $\mathcal{M}$ and $\mathcal{M}'$ are monoidal categories. Since $H' = H/I$ is a quotient Hopf algebra of H, there is a monoidal category embedding functor from $\mathcal{M}'$ to $\mathcal{M}$. Thus, one can regards $\mathcal{M}'$ as a monoidal full subcategory of $\mathcal{M}$. Moreover, an object M in $\mathcal{M}$ is an object in the subcategory $\mathcal{M}'$ if and only if $I \cdot M = 0$. We present the observation as follows.

Theorem 4.1. $\mathcal{M}'$ *is a monoidal full subcategory of* $\mathcal{M}$ *consisting of all those H-modules M such that* $I \cdot M = 0$.

From now on, we assume that $\chi^{-1}(a) \neq 1$, i.e. H is of **Case 1** (or H' is of **Type 1**). In this case, χ is not the trivial character. We may assume that $\delta = 0$ and $H = kG(\chi^{-1}, a, 0)$. That is, H is generated, as an algebra, by G and x subject to the relation: $xg = \chi^{-1}(g)gx$, $g \in G$. The coalgebra structure and antipode are given by

$$\Delta(x) = x \otimes a + 1 \otimes x, \quad S(x) = -xa^{-1}, \quad \varepsilon(x) = 0,$$
$$\Delta(g) = g \otimes g, \quad S(g) = g^{-1}, \quad \varepsilon(g) = 1, \quad g \in G.$$

H has a k-basis $\{gx^i \mid g \in G, i \geq 0\}$. Let $H(n) = \mathrm{span}\{gx^n \mid g \in G\}$ for all $n \geq 0$. Then $H = \bigoplus_{n=0}^{\infty} H(n)$ is a graded coalgebra, and $H(0) = H_0$ is the coradical of H by Proposition 2.2. Obviously, H is also a graded algebra and $S(H(n)) \subseteq H(n)$ for all $n \geq 0$. Thus, H is a graded Hopf algebra (see [4] for the definition of graded Hopf algebra).

4.1. **Simple modules.** Let V be an H-module. Then V becomes an H_0-module by the restriction. Conversely, for a kG-module W, W becomes an H-module by setting $x \cdot w = 0$, $w \in W$. Moreover, W is a graded H-module with the trivial grading given by $W_{(0)} = W$ and $W_{(i)} = 0$ for $i > 0$.

Let $V = \oplus_{i=0}^{\infty} V_{(i)}$ be a graded H-module. Then $g \cdot V_{(i)} = V_{(i)}$ and $x \cdot V_{(i)} \subseteq V_{(i+1)}$ for all $g \in G$ and $i \geq 0$. Let $V_{[n]} = V_{(n)} \oplus V_{(n+1)} \oplus V_{(n+2)} \oplus \cdots = \oplus_{i=n}^{\infty} V_{(i)}$. Then $V = V_{[0]} \supseteq V_{[1]} \supseteq V_{[2]} \supseteq \cdots$ is a descending sequence of graded H-submodules. If V is simple, then there exists some $n \geq 0$ such that $V = V_{[0]} = V_{[1]} = V_{[2]} = \cdots = V_{[n]} = V_{(n)}$, and $V_{[n+1]} = 0$. Hence $x \cdot V = 0$ in this case. Thus, we have the following proposition.

Lemma 4.2. *Let V be a graded H-module. Then V is simple if and only if V is simple as an H_0-module.*

Proof. Assume that V is simple as an H-module. Then $x \cdot V = 0$ from the discussion above. Hence each H_0-submodule of V is an H-submodule. It follows that V is also simple as an H_0-module. The converse is obvious. □

Note that a graded module over a graded algebra is graded simple if it is nonzero and has no nontrivial graded submodule.

Proposition 4.3. *Let $A = \oplus_{i=0}^{\infty} A_i$ be a graded algebra, and $M = \oplus_{i=0}^{\infty} M_i$ a graded A-module. Then M is graded simple if and only if M is simple.*

Proof. If M is simple, then M is obviously graded simple. Conversely, if $M = \oplus M_{(i)}$ is graded simple, then for any $n \geq 0$, $M(n) := \oplus_{i=n}^{\infty} M_i$ is a graded A-submodule of M and $M = M(0) \supseteq M(1) \supseteq M(2) \supseteq \cdots$. Hence there exists some $n \geq 0$ such that $M = M(1) = \cdots = M(n)$ and $M(n+1) = 0$, and hence $M_n \neq 0$ and $M_i = 0$ for $i \neq n$. Thus $M = M_n$ is simple. □

Let M be an H-module. For any $\lambda \in \hat{G}$, let $M_{(\lambda)} = \{v \in M \mid g \cdot v = \lambda(g)v, g \in G\}$. Each nonzero element in $M_{(\lambda)}$ is called a weight vector of weight λ in M. One can check that $\oplus_{\lambda \in \hat{G}} M_{(\lambda)}$ is a submodule of M.

Let $\Pi(M) = \{\lambda \in \hat{G} \mid M_{(\lambda)} \neq 0\}$, which is called the weight space of M. M is said to be a weight module if $M = \oplus_{\lambda \in \Pi(M)} M_{(\lambda)}$. If G is abelian and k is algebraically closed, then $\Pi(M)$ is not empty for any nonzero finite dimensional H-module M. Let $\mathcal{W}$ be the full subcategory of $\mathcal{M}$ consisting of all weight modules. Similarly, one can define weight modules over H'. Let $\mathcal{W}'$ be the full subcategory of $\mathcal{M}'$ consisting of all weight modules. Then one can easily see that $\mathcal{W}$ (resp. $\mathcal{W}'$) is a monoidal full subcategory of $\mathcal{M}$ (resp. $\mathcal{M}'$). Moreover, $\mathcal{W}' = \mathcal{W} \cap \mathcal{M}'$.

For any $\lambda \in \hat{G}$, let V_λ be the 1-dimensional H-module defined by

$$g \cdot v = \lambda(g)v \text{ and } x \cdot v = 0,\ v \in V_\lambda.$$

The following lemma is obvious.

Lemma 4.4. *Let $\sigma, \lambda \in \hat{G}$. Then*

(a) *V_λ is a simple H-module.*

(b) *$V_\sigma \cong V_\lambda$ if and only if $\sigma = \lambda$.*

(c) *$V_\sigma \otimes V_\lambda \cong V_{\sigma\lambda}$.*

When $|\chi| = s < \infty$, for any $\lambda \in \hat{G}$ and $\beta \in k$, let $V(\lambda,\beta)$ be an s-dimensional vector space with a basis $\{m_0, m_1, \cdots, m_{s-1}\}$ over k. Then one can easily check that $V(\lambda,\beta)$ is an H-module with the action determined by

$$g \cdot m_i = \chi^i(g)\lambda(g)m_i, \quad x \cdot m_i = \begin{cases} m_{i+1}, & 0 \le i \le s-2 \\ \beta m_0, & i = s-1 \end{cases},$$

where $0 \le i \le s-1$, $g \in G$. Moreover, $x^s \cdot m = \beta m$ for all $m \in V(\lambda,\beta)$. In this case, let $\langle\chi\rangle$ denote the subgroup of $\hat{G}$ generated by χ, and $[\lambda]$ denote the image of λ under the canonical epimorphism $\hat{G} \to \hat{G}/\langle\chi\rangle$.

Theorem 4.5. *Assume $|\chi| = s < \infty$. Let $\lambda \in \hat{G}$ and $\beta \in k$. Then*

(a) *$V(\lambda,\beta)$ is indecomposable.*

(b) *$V(\lambda, 0)$ is not simple but uniserial.*

(c) *If $\beta \neq 0$, then $V(\lambda,\beta)$ is simple.*

Proof. Note that $s > 1$. Let $\{m_0, m_1, \cdots, m_{s-1}\}$ be the basis of $V(\lambda,\beta)$ as described above. Then $V(\lambda,\beta)$ is a cyclic H-module generated by m_0.

(a) Suppose that $V(\lambda,\beta) = U_1 \oplus U_2$ for some submodules U_1, U_2 of $V(\lambda,\beta)$. Then $m_0 = u_1 + u_2$ for some $u_1 \in U_1$ and $u_2 \in U_2$. For any $g \in G$, we have $g \cdot m_0 = \lambda(g)m_0 = \lambda(g)u_1 + \lambda(g)u_2$ and $g \cdot m_0 = g \cdot u_1 + g \cdot u_2$. Hence $g \cdot u_1 = \lambda(g)u_1$ and $g \cdot u_2 = \lambda(g)u_2$ for all $g \in G$, which implies that $u_1, u_2 \in V(\lambda,\beta)_{(\lambda)}$. Since $|\chi| = s$, $V(\lambda,\beta)_{(\lambda)}$ is one dimensional with a basis $\{m_0\}$, which implies that $u_1 = 0$ or $u_2 = 0$. If $u_1 = 0$, then $m_0 = u_2 \in U_2$, and so $U_2 = V(\lambda,\beta)$ and $U_1 = 0$. Similarly, if $u_2 = 0$, then $U_1 = V(\lambda,\beta)$ and $U_2 = 0$. Therefore, $V(\lambda,\beta)$ is an indecomposable H-module.

(b) Let $\beta = 0$ in the above. Then km_{s-1} is a 1-dimensional submodule of $V(\lambda, 0)$ and $km_{s-1} \cong V_{\chi^{s-1}\lambda}$. Hence $V(\lambda, 0)$ is not simple since $\dim(V(\lambda, 0)) = s > 1$. For any $0 \le t \le s-1$, one can easily see that $M_t = \mathrm{span}\{m_i \mid t \le i \le s-1\}$ is a submodule of $V(\lambda, 0)$ with dimension $s-t$. Obviously,

$$V(\lambda, 0) = M_0 \supset M_1 \supset M_2 \supset \cdots \supset M_{s-1} \supset M_s = 0$$

is a composition series of $V(\lambda, 0)$ and $M_t/M_{t+1} \cong V_{\chi^t\lambda}$ for all $0 \le t \le s-1$.

Now let M be a non-zero submodule of $V(\lambda, 0)$. Then $x^s \cdot M \subseteq x^s \cdot V(\lambda, 0) = 0$. Hence there is an integer t with $1 \le t \le s$ such that $x^t \cdot M = 0$ but $x^{t-1} \cdot M \neq 0$. If $t = s$, then $M \subseteq V(\lambda, 0) = M_0 = M_{s-t}$. If $t < s$ and $m = \alpha_0 m_0 + \alpha_1 m_1 + \cdots + \alpha_{s-1}m_{s-1} \in M$, then $x^t \cdot m = \alpha_0 m_t + \alpha_1 m_{t+1} + \cdot + \alpha_{s-t-1}m_{s-1} = 0$ since $x^t \cdot M = 0$, which implies that

$\alpha_0 = \alpha_1 = \cdots = \alpha_{s-t-1} = 0$. Hence $m = \alpha_{s-t}m_{s-t}+\cdots+\alpha_{s-1}m_{s-1} \in M_{s-t}$, and so $M \subseteq M_{s-t}$. Thus, we have proven that $M \subseteq M_{s-t}$. Since $x^{t-1}\cdot M \neq 0$, one can choose an element $m \in M$ such that $x^{t-1}\cdot m \neq 0$. From $M \subseteq M_{s-t}$, we have $m = \alpha_{s-t}m_{s-t} + \cdots + \alpha_{s-1}m_{s-1}$ for some $\alpha_{s-t}, \cdots, \alpha_{s-1} \in k$. From $x^{t-1}\cdot m = \alpha_{s-t}m_{s-1} \neq 0$, one gets that $\alpha_{s-t} \neq 0$. Now we have

$$\begin{array}{lcr} m & = & \alpha_{s-t}m_{s-t} + \alpha_{s-t+1}m_{s-t+1}\cdots + \alpha_{s-1}m_{s-1} \\ x\cdot m & = & \alpha_{s-t}m_{s-t+1}\cdots + \alpha_{s-2}m_{s-1} \\ & & \cdots \\ x^{t-1}\cdot m & = & \alpha_{s-t}m_{s-1} \end{array}$$

Since $\alpha_{s-t} \neq 0$ and $m, x\cdot m, \cdots, x^{t-1}\cdot m \in M$, one knows that $m_{s-t}, m_{s-t+1}, \cdots, m_{s-1} \in M$, and so $M_{s-t} \subseteq M$. Thus, $M = M_{s-t}$. It follows that $V(\lambda, 0)$ is a uniserial H-module.

(c) Assume $\beta \neq 0$. Then $V(\lambda,\beta) = V(\lambda,\beta)_{(\lambda)} \oplus V(\lambda,\beta)_{(\chi\lambda)} \oplus \cdots \oplus V(\lambda,\beta)_{(\chi^{s-1}\lambda)}$ and $V(\lambda,\beta)_{(\chi^i\lambda)} = km_i$ for all $0 \leq i \leq s-1$. Moreover, for any $0 \leq i \leq s-1$, the submodule of $V(\lambda,\beta)$ generated by m_i is equal to $V(\lambda,\beta)$.

Now let M be a nonzero submodule of $V(\lambda,\beta)$. Since $V(\lambda,\beta)$ is a weight module, so is M. Hence $\Pi(M)$ is not empty. Since $\Pi(M) \subseteq \Pi(V(\lambda,\beta)) = \{\chi^i\lambda \mid 0 \leq i \leq s-1\}$, there is an i with $0 \leq i \leq s-1$ such that $\chi^i\lambda \in \Pi(M)$, and so $0 \neq M_{(\chi^i\lambda)} \subseteq V(\lambda,\beta)_{(\chi^i\lambda)} = km_i$. It follows that $m_i \in M$, which forces that $M = V(\lambda,\beta)$. Therefore, $V(\lambda,\beta)$ is simple. □

Proposition 4.6. *Assume $|\chi| = s < \infty$. Let $\sigma, \lambda \in \hat{G}$ and $\alpha, \beta \in k^\times$. Then*

(a) *$V(\sigma, 0) \ncong V(\lambda,\beta)$.*

(b) *$V(\sigma, 0) \cong V(\lambda, 0)$ if and only if $\sigma = \lambda$.*

(c) *$V(\sigma,\alpha) \cong V(\lambda,\beta)$ if and only if $[\sigma] = [\lambda]$ and $\alpha = \beta$.*

Proof. (a) By Theorem 4.5, $V(\lambda,\beta)$ is simple, but $V(\sigma, 0)$ is not simple. Hence $V(\sigma, 0) \ncong V(\lambda,\beta)$.

(b) If $\sigma = \lambda$, then obviously $V(\sigma, 0) \cong V(\lambda, 0)$. Conversely, let $f : V(\sigma, 0) \to V(\lambda, 0)$ be an H-module isomorphism. Let $U = \{v \in V(\sigma, 0) \mid x\cdot v = 0\}$ and $W = \{v \in V(\lambda, 0) \mid x\cdot v = 0\}$. Then $\dim(U) = \dim(W) = 1$ and $f(U) = W$. Moreover, $g\cdot u = \chi^{s-1}(g)\sigma(g)u$ and $g\cdot w = \chi^{s-1}(g)\lambda(g)w$ for any $g \in G$, $u \in U$ and $w \in W$. Let $0 \neq u \in U$. Then $f(g\cdot u) = g\cdot f(u)$, and so $\chi^{s-1}(g)\sigma(g) = \chi^{s-1}(g)\lambda(g)$ for all $g \in G$. This implies that $\sigma = \lambda$.

(c) Assume $V(\sigma,\alpha) \cong V(\lambda,\beta)$. Then $\alpha = \beta$ since $x^s\cdot u = \alpha u$ and $x^s\cdot v = \beta v$ for any $u \in V(\sigma,\alpha)$ and $v \in V(\lambda,\beta)$. Moreover, $\Pi(V(\sigma,\alpha)) = \Pi(V(\lambda,\beta))$. Since $\Pi(V(\sigma,\alpha)) = \{\chi^i\sigma | 0 \leq i \leq s-1\}$ and $\Pi(V(\lambda,\beta)) = \{\chi^i\lambda | 0 \leq i \leq s-1\}$, $\sigma = \chi^i\lambda$ for some $0 \leq i \leq s-1$. Hence $[\sigma] = [\lambda]$.

Conversely, assume $\alpha = \beta$ and $[\sigma] = [\lambda]$. Then $\sigma = \chi^i\lambda$ for some $0 \leq i \leq s-1$ since $|\chi| = s$. Let $\{m_0, m_1, \cdots, m_{s-1}\}$ and $\{v_0, v_1, \cdots, v_{s-1}\}$ be the bases of $V(\sigma,\alpha)$ and $V(\lambda,\beta)$ as before, respectively. Define a k-linear isomorphism $f : V(\sigma,\alpha) \to V(\lambda,\beta)$ by $f(m_0) = v_i$, $f(m_1) = v_{i+1}, \cdots, f(m_{s-i-1}) = v_{s-1}$, $f(m_{s-i}) = \beta v_0, \cdots, f(m_{s-1}) = \beta v_{i-1}$. Then one can easily check that f is an H-module map. Hence $V(\sigma,\alpha) \cong V(\lambda,\beta)$. □

Proposition 4.7. *Assume $|\chi| = s < \infty$. Let $\sigma, \lambda \in \hat{G}$ and $\alpha \in k$. Then $V_\lambda \otimes V(\sigma,\alpha) \cong V(\lambda\sigma,\alpha)$ and $V(\sigma,\alpha) \otimes V_\lambda \cong V(\sigma\lambda, \alpha\lambda(a)^s)$.*

Proof. It follows from a straightforward verification. $\square$

Proposition 4.8. *Assume that $|\chi| = s < \infty$ and that $\chi^{-1}(a)$ is a primitive s-th root of unity. Let $\sigma, \lambda \in \hat{G}$ and $\alpha, \beta \in k$. Then $V(\sigma,\alpha) \otimes V(\lambda,\beta) \cong \bigoplus_{t=0}^{s-1} V(\chi^t\sigma\lambda, \alpha\lambda(a)^s + \beta)$.*

Proof. Let $M = V(\sigma,\alpha)\otimes V(\lambda,\beta)$. Let $\{m_0, m_1, \cdots, m_{s-1}\}$ and $\{v_0, v_1, \cdots, v_{s-1}\}$ be the bases of $V(\sigma,\alpha)$ and $V(\lambda,\beta)$ as described before, respectively. Then $\{m_i \otimes v_j | 0 \leq i, j \leq s-1\}$ is a k-basis of M. For any $0 \leq i, j \leq s-1$ and $g \in G$, we have

$$g \cdot (m_i \otimes v_j) = g \cdot m_i \otimes g \cdot v_j = (\chi^{i+j}\sigma\lambda)(g) m_i \otimes v_j,$$

and hence $m_i \otimes v_j \in M_{(\chi^{i+j}\sigma\lambda)}$. Thus, M is a weight module. Since $|\chi| = s$, $M = M_{(\sigma\lambda)} \oplus M_{(\chi\sigma\lambda)} \oplus \cdot \oplus M_{(\chi^{s-1}\sigma\lambda)}$, and for any $0 \leq t \leq s-1$,

$$M_{(\chi^t\sigma\lambda)} = \mathrm{span}\{m_i \otimes v_{t-i}, m_j \otimes v_{s+t-j} \mid 0 \leq i \leq t, t+1 \leq j \leq s-1\}.$$

Thus, $\dim(M_{(\chi^t\sigma\lambda)}) = s$ for all $0 \leq t \leq s-1$. Moreover, $x \cdot M_{(\chi^t\sigma\lambda)} \subseteq M_{(\chi^{t+1}\sigma\lambda)}$ for all $0 \leq t \leq s-1$, where $M_{(\chi^s\sigma\lambda)} = M_{(\sigma\lambda)}$ since $|\chi| = s$. Since $q := \chi^{-1}(a)$ is a primitive s-th root of unity and $(1 \otimes x)(x \otimes a) = q(x \otimes a)(1 \otimes x)$, we have $\Delta(x^s) = x^s \otimes a^s + 1 \otimes x^s$. Now for any $0 \leq i, j \leq s-1$, we have

$$\begin{aligned}
x^s \cdot (m_i \otimes v_j) &= x^s \cdot m_i \otimes a^s \cdot v_j + m_i \otimes x^s \cdot v_j \\
&= \alpha\chi^j(a^s)\lambda(a^s) m_i \otimes v_j + \beta m_i \otimes v_j \\
&= (\alpha q^{-js}\lambda(a)^s + \beta) m_i \otimes v_j \\
&= (\alpha\lambda(a)^s + \beta) m_i \otimes v_j.
\end{aligned}$$

Hence $x^s \cdot m = (\alpha\lambda(a)^s + \beta)m$ for all $m \in M$.

(a) Assume $\alpha\lambda(a)^s + \beta \neq 0$. Then $x\cdot : M \to M$, $m \mapsto x \cdot m$, is a linear automorphism of M. Let $\{u_1, u_2, \cdots, u_s\}$ be a basis of $M_{(\sigma\lambda)}$. Then $\{x^t \cdot u_1, x^t \cdot u_2, \cdots, x^t \cdot u_s\}$ is a basis of $M_{(\chi^t\sigma\lambda)}$ for all $0 \leq t \leq s-1$. It follows that $\{x^t \cdot u_i \mid 0 \leq t \leq s-1, 1 \leq i \leq s\}$ is a basis of M. Let $U_i = \mathrm{span}\{x^t \cdot u_i | 0 \leq t \leq s-1\}$ for any $1 \leq i \leq s$. Then $M = U_1 \oplus U_2 \oplus \cdots \oplus U_s$. It is easy to see that U_i is a submodule of M and $U_i \cong V(\sigma\lambda, \alpha\lambda(a)^s + \beta)$ for all $1 \leq i \leq s$. By Proposition 4.6(c), $V(\chi^t\sigma\lambda, \alpha\lambda(a)^s + \beta) \cong V(\sigma\lambda, \alpha\lambda(a)^s + \beta)$ for all $0 \leq t \leq s-1$. It follows that $V(\sigma,\alpha) \otimes V(\lambda,\beta) \cong \bigoplus_{t=0}^{s-1} V(\chi^t\sigma\lambda, \alpha\lambda(a)^s + \beta)$ in this case.

(b) Assume $\alpha\lambda(a)^s + \beta = 0$. Let $f : M \to M$ be the linear endomorphism of M defined by $f(m) = x \cdot m$, $m \in M$. Then $f^s = 0$. Let $m \in M_{(\sigma\lambda)}$. Then

$$m = \alpha_0 m_0 \otimes v_0 + \alpha_1 m_1 \otimes v_{s-1} + \alpha_2 m_2 \otimes v_{s-2} + \cdots + \alpha_{s-1} m_{s-1} \otimes v_1$$

for some $\alpha_0, \alpha_1, \cdots, \alpha_{s-1} \in k$. Hence, we have

$$\begin{array}{rcl} x \cdot m & = & \alpha_0 x \cdot (m_0 \otimes v_0) + \sum\limits_{i=1}^{s-1} \alpha_i x \cdot (m_i \otimes v_{s-i}) \\ & = & \alpha_0(x \cdot m_0 \otimes a \cdot v_0 + m_0 \otimes x \cdot v_0) + \sum\limits_{i=1}^{s-1} \alpha_i (x \cdot m_i \otimes a \cdot v_{s-i} + m_i \otimes x \cdot v_{s-i}) \\ & = & \alpha_0(m_1 \otimes \lambda(a) v_0 + m_0 \otimes v_1) + \alpha_{s-1} \alpha m_0 \otimes \chi(a)\lambda(a) v_1 + \\ & & + \sum\limits_{i=1}^{s-2} \alpha_i m_{i+1} \otimes \chi^{s-i}(a)\lambda(a) v_{s-i} + \sum\limits_{i=2}^{s-1} \alpha_i m_i \otimes v_{s-i+1} + \alpha_1 \beta m_1 \otimes v_0 \\ & = & (\alpha_0 + \alpha\lambda(a) q^{s-1} \alpha_{s-1}) m_0 \otimes v_1 + (\lambda(a)\alpha_0 + \beta\alpha_1) m_1 \otimes v_0 + \\ & & + \sum\limits_{i=2}^{s-1} (q^{i-1}\lambda(a)\alpha_{i-1} + \alpha_i) m_i \otimes v_{s+1-i}. \end{array}$$

Thus, $x \cdot m = 0$ if and only if $(\alpha_0, \alpha_1, \cdots, \alpha_{s-1})$ is a solution of the following system of linear equations

$$\left\{ \begin{array}{rcl} x_0 + \alpha\lambda(a) q^{s-1} x_{s-1} & = & 0 \\ \lambda(a) x_0 + \beta x_1 & = & 0 \\ \lambda(a) q x_1 + x_2 & = & 0 \\ \lambda(a) q^2 x_2 + x_3 & = & 0 \\ \cdots\cdots & & \\ \lambda(a) q^{s-2} x_{s-2} + x_{s-1} & = & 0 \end{array} \right.$$

Since q is a primitive s-th root of unity and $\alpha\lambda(a)^s + \beta = 0$, one can easily check that the rank of the coefficient matrix of the above system of linear equations is $s-1$. It follows that $\mathrm{Ker}(f) \cap M_{(\sigma\lambda)}$ is a 1-dimensional subspace of $M_{(\sigma\lambda)}$. Similarly, one can show that $\dim(\mathrm{Ker}(f) \cap M_{(\chi^t\sigma\lambda)}) = 1$ for all $1 \le t \le s-1$. Since $x \cdot M_{(\chi^t\sigma\lambda)} \subseteq M_{(\chi^{t+1}\sigma\lambda)}$, we have

$$\mathrm{Ker}(f) = (\mathrm{Ker}(f) \cap M_{(\sigma\lambda)}) \oplus (\mathrm{Ker}(f) \cap M_{(\chi\sigma\lambda)}) \oplus \cdots \oplus (\mathrm{Ker}(f) \cap M_{(\chi^{s-1}\sigma\lambda)})$$

and so $\dim(\mathrm{Ker}(f)) = s$. Thus, $\dim(\mathrm{Im}(f^i)) = \dim(\mathrm{Im}(f^{i-1})) - \dim(\mathrm{Im}(f^{i-1}) \cap \mathrm{Ker}(f)) \ge \dim(\mathrm{Im}(f^{i-1})) - s$, which implies that $\dim(\mathrm{Im}(f^i)) \ge \dim(\mathrm{Im}(f^0)) - is = s^2 - is = (s-i)s$, where $i \ge 1$. In particular, $\dim(\mathrm{Im}(f^{s-1})) \ge s$. Since $f^s = 0$, $\mathrm{Im}(f^{s-1}) \subseteq \mathrm{Ker}(f)$, and so $\dim(\mathrm{Im}(f^{s-1})) \le \dim(\mathrm{Ker}(f)) = s$. It follows that $\dim(\mathrm{Im}(f^{s-1})) = \dim(\mathrm{Ker}(f)) = s$. Consequently, $\mathrm{Im}(f^{s-1}) = \mathrm{Ker}(f)$, and $\dim(\mathrm{Im}(f^{s-1}) \cap M_{(\chi^t\sigma\lambda)}) = \dim(\mathrm{Ker}(f) \cap M_{(\chi^t\sigma\lambda)}) = 1$ for any $t \in \mathbb{Z}$. Since $\mathrm{Im}(f^{s-1}) \cap M_{(\chi^t\sigma\lambda)} \subseteq f^{s-1}(M_{(\chi^{t+1}\sigma\lambda)})$ for any $t \in \mathbb{Z}$, it follows that $f^{s-1}(M_{(\chi^t\sigma\lambda)}) \neq 0$ for any $t \in \mathbb{Z}$. Thus, for any $0 \le t \le s-1$, one can choose an element $u_t \in M_{(\chi^t\sigma\lambda)}$ such that $f^{s-1}(u_t) \neq 0$. Now let $0 \le t \le s-1$. Then $u_t, f(u_{t-1}), \cdots, f^t(u_0), f^{t+1}(u_{s-1}), \cdots, f^{s-1}(u_{t+1}) \in M_{(\chi^t\sigma\lambda)}$. From $f^{s-1}(u_i) \neq 0$ and $f^s(u_i) = 0$ for all $0 \le i \le s-1$, one knows that the following elements

$$u_t, f(u_{t-1}), \cdots, f^t(u_0), f^{t+1}(u_{s-1}), \cdots, f^{s-1}(u_{t+1})$$

are linearly independent over k, and hence form a k-basis of $M_{(\chi^t\sigma\lambda)}$. Consequently, $\{f^i(u_t) \mid 0 \le i, t \le s-1\}$ is a k-basis of M. Let $U_t = \mathrm{span}\{f^i(u_t) \mid 0 \le i \le s-1\}$ for any $0 \le t \le s-1$. Then $M = U_0 \oplus U_1 \oplus \cdots \oplus U_{s-1}$. One can easily check that U_t is a submodule of M and $U_t \cong V(\chi^t\sigma\lambda, 0)$ for any $0 \le t \le s-1$. This completes the proof. □

Throughout the rest of this section, assume that G is an abelian group and k is an algebraically closed field.

Let M be a finite dimensional H-module. Since k is an algebraically closed field and kG is a commutative algebra, there is a $\lambda \in \hat{G}$ such that $M_{(\lambda)} \neq 0$. Hence $\oplus_{\lambda \in \hat{G}} M_{(\lambda)} =$

$\oplus_{\lambda\in\Pi(M)}M_{(\lambda)}$ is a nonzero submodule of M. Thus, if M is a simple H-module, then $M = \oplus_{\lambda\in\Pi(M)}M_{(\lambda)}$, which implies that M is a weight module.

Now assume that M is a finite dimensional simple H-module. Then $\Pi(M) = \{\lambda \in \hat{G} \mid M_{(\lambda)} \neq 0\}$ is a non-empty finite set. If $|\chi| = \infty$, there exists a $\lambda \in \Pi(M)$ such that $\chi\lambda \notin \Pi(M)$. Thus, $x \cdot M_{(\lambda)} = 0$, and it follows that $M = M_{(\lambda)} \cong V_\lambda$.

Now suppose $|\chi| = s < \infty$. Let $M_0 = \{m \in M \mid x \cdot m = 0\}$. Then one can easily check that M_0 is a submodule of M. Hence, if $M_0 \neq 0$, then $M = M_0$ since M is simple. Since $\Pi(M)$ is not empty, M is isomorphic to some V_λ. If $M_0 = 0$, then the induced map $x\cdot : M \to M$, $m \mapsto x \cdot m$, is a bijective map. Pick up a $\lambda \in \Pi(M)$. Then

$$x \cdot M_{(\lambda)} \subseteq M_{(\chi\lambda)}, \cdots, x^{s-1} \cdot M_{(\lambda)} \subseteq M_{(\chi^{s-1}\lambda)}, x^s \cdot M_{(\lambda)} \subseteq M_{(\chi^s\lambda)} = M_{(\lambda)}.$$

Thus $M_{(\lambda)} \oplus M_{(\chi\lambda)} \oplus \cdots \oplus M_{(\chi^{s-1}\lambda)}$ is a nonzero submodule of M. Since M is simple, one gets that $M = M_{(\lambda)} \oplus M_{(\chi\lambda)} \oplus \cdots \oplus M_{(\chi^{s-1}\lambda)}$. Moreover, $\dim(M_{(\lambda)}) = \dim(M_{(\chi\lambda)}) = \cdots = \dim(M_{(\chi^{s-1}\lambda)})$. Considering the action of x^s on $M_{(\lambda)}$, one gets a linear automorphism $x^s\cdot : M_{(\lambda)} \to M_{(\lambda)}$, $m \mapsto x^s \cdot m$. Since k is algebraically closed, there is a nonzero element $m \in M_{(\lambda)}$ such that $x^s \cdot m = \beta m$ with $\beta \in k^\times$. Then it is easy to see that $\mathrm{span}\{m, x \cdot m, \cdots, x^{s-1} \cdot m\}$ is a submodule of M. Since M is simple, $M = \mathrm{span}\{m, x \cdot m, \cdots, x^{s-1} \cdot m\}$. Obviously $M \cong V(\lambda,\beta)$. Thus we have the following theorem.

Theorem 4.9. *Let M be a finite dimensional simple H-module. Then M is a weight module. Furthermore, we have:*

(a) *If $|\chi| = \infty$, then $M \cong V_\lambda$ for some $\lambda \in \hat{G}$.*

(b) *If $|\chi| = s < \infty$, then either $M \cong V_\lambda$ for some $\lambda \in \hat{G}$, or $M \cong V(\lambda,\beta)$ for some $\lambda \in \hat{G}$ and $\beta \in k^\times$.*

Since $H' \cong H/I$ as a Hopf algebra, a vector space M over k is an H'-module (a simple H'-module) if and only if M is an H-module (a simple H-module) such that $I \cdot M = 0$, where I is the Hopf ideal of H as described in Theorem 3.4. Note that $\chi^{-1}(a) \neq 1$, and hence $|\chi| > 1$. It follows from Theorem 3.4 that $I = 0$ if and only if $\chi^{-1}(a)$ is not a primitive n-th root of unity for any $n \geq 2$. Moreover, $|\chi| = \infty$ if $I = 0$. If $\chi^{-1}(a)$ is a primitive n-th root of unity for some $n \geq 2$, then by Corollary 3.6, one knows that $I = \langle x^n\rangle$, or $I = \langle x^n, 1 - a^n\rangle$ with $a^n \neq 1$ and $n < |\chi|$, or $I = \langle x^n - \beta(1 - a^n)\rangle$ with $\beta \neq 0$, $a^n \neq 1$ and $|\chi| = n$. Thus, as a corollary of Theorem 4.9, one can get all the finite dimensional simple H'-modules.

Corollary 4.10. *If $\chi^{-1}(a)$ is not a primitive n-th root of unity for any $n \geq 2$, then $\mathcal{S} = \{V_\lambda \mid \lambda \in \hat{G}\}$ is a complete set of finite dimensional simple H'-modules.*

Corollary 4.11. *Assume that $\chi^{-1}(a)$ is a primitive n-th root of unity for some $n \geq 2$. Then a complete set $\mathcal{S}$ of finite dimensional simple H'-modules can be described as follows:*

(a) *If $I = \langle x^n\rangle$, then $\mathcal{S} = \{V_\lambda \mid \lambda \in \hat{G}\}$.*

(b) *If $I = \langle x^n, 1 - a^n\rangle$ with $a^n \neq 1$ and $n < |\chi|$, then $\mathcal{S} = \{V_\lambda \mid \lambda \in \hat{G}$ with $\lambda(a)^n = 1\}$.*

(c) *If $I = \langle x^n - \beta(1 - a^n)\rangle$ with $\beta \in k^\times$, $a^n \neq 1$ and $|\chi| = n$, then*

$$\mathcal{S} = \{V_\lambda, V(\sigma, \beta(1 - \sigma(a)^n)) \mid \lambda, \sigma \in \hat{G} \text{ with } \lambda(a)^n = 1, \sigma(a)^n \neq 1\}.$$

Moreover, for any $\lambda, \sigma \in \hat{G}$ with $\lambda(a)^n \neq 1$ and $\sigma(a)^n \neq 1$, $V(\lambda, \beta(1 - \lambda(a)^n)) \cong V(\sigma, \beta(1 - \sigma(a)^n))$ if and only if $[\lambda] = [\sigma]$.

Proof. Parts (a) and (b) can be shown easily. Now we show Part (c). Assume $I = \langle x^n - \beta(1-a^n)\rangle$ with $\beta \in k^\times$, $a^n \neq 1$ and $|\chi| = n$. Since $|\chi| = n < \infty$, each finite dimensional simple H-module is of the form V_λ or $V(\sigma,\gamma)$ for some $\lambda,\sigma \in \hat{G}$ and $\gamma \in k^\times$ by Theorem 4.9(b). Let $0 \neq v \in V_\lambda$. Then $V_\lambda = kv$, $g \cdot v = \lambda(g)v$ and $x \cdot v = 0$ for all $g \in G$. Hence $(x^n - \beta(1-a^n)) \cdot v = -\beta(1-\lambda(a)^n)v$. Thus, $(x^n - \beta(1-a^n)) \cdot V_\lambda = 0$ if and only if $\lambda(a)^n = 1$. For $V(\sigma,\gamma)$, since $|\chi| = n$, $V(\sigma,\gamma) = \text{span}\{m, x \cdot m, \cdots, x^{n-1} \cdot m\}$ such that $g \cdot m = \sigma(g)m$ for any $g \in G$ and $x^n \cdot m = \gamma m$. Since $(x^n - \beta(1-a^n))x = x(x^n - \beta(1-a^n))$, $V(\sigma,\gamma)$ is an H'-module if and only if $(x^n - \beta(1-a^n)) \cdot V(\sigma,\gamma) = 0$ if and only if $(x^n - \beta(1-a^n)) \cdot m = 0$ if and only if $\gamma = \beta(1-\sigma(a)^n)$. This completes the proof of Part (c).

If $[\lambda] = [\sigma]$, then $\lambda(a)^n = \sigma(a)^n$ by $|\chi| = n$ or the assumption that $\chi^{-1}(a)$ is a primitive n-th root of unity. It follows from Lemma 4.4(c) that $V(\lambda,\beta(1-\lambda(a)^n)) \cong V(\sigma,\beta(1-\sigma(a)^n))$ if and only if $[\lambda] = [\sigma]$. □

4.2. **Verma modules.** For each $\lambda \in \hat{G}$, one can define a left H-module $M(\lambda) = H \otimes_{H_0} V_\lambda$, called a Verma module. Note that H is a free right H_0-module and $H = \bigoplus_{i=0}^{\infty}(x^i H_0)$ as right H_0-modules, one gets that

$$x^i H_0 \cong H_0, \quad M(\lambda) = \bigoplus_{i=0}^{\infty}(x^i H_0) \otimes_{H_0} V_\lambda = \bigoplus_{i=0}^{\infty} x^i \otimes_{H_0} V_\lambda.$$

For the sake of simplicity, we write v_λ for $1 \otimes_{H_0} v_\lambda$ with $0 \neq v_\lambda \in V_\lambda$. Then $x^i \otimes_{H_0} v_\lambda = x^i \cdot (1 \otimes_{H_0} v_\lambda) = x^i \cdot v_\lambda$. Hence $M(\lambda)$ has a k-basis $\{x^i \cdot v_\lambda \mid i \geq 0\}$, and so $M(\lambda)$ is a free $k[x]$-module of rank one. Moreover, $M(\lambda) = k[x] \cdot v_\lambda$, where $k[x]$ is the subalgebra of H generated by x, which is a polynomial algebra in one variable x over k.

Let $J(\lambda) := \text{span}\{x^i \cdot v_\lambda \mid i > 0\}$. Then $J(\lambda)$ is a submodule of $M(\lambda)$. Denote by $L(\lambda)$ the corresponding quotient module $M(\lambda)/J(\lambda)$. That is, $L(\lambda) = M(\lambda)/J(\lambda)$.

When $|\chi| = s < \infty$, let $J_\beta(\lambda)$ be the submodule of $M(\lambda)$ generated by $(x^s - \beta) \cdot v_\lambda$, and $L(\lambda,\beta) := M(\lambda)/J_\beta(\lambda)$ be the corresponding quotient module, where $\beta \in k$.

Theorem 4.12. *Let $\lambda, \tau \in \hat{G}$. Then*

(a) *$M(\lambda)$ is a weight module.*

(b) *$M(\lambda)$ is indecomposable.*

(c) *$J(\lambda)$ is a maximal submodule of $M(\lambda)$ and $L(\lambda) \cong V_\lambda$.*

(d) *$M(\lambda) \cong M(\tau)$ if and only if $\lambda = \tau$.*

(e) *If $|\chi| = \infty$, then $J(\lambda)$ is the unique maximal submodule of $M(\lambda)$.*

(f) *Assume $|\chi| = s < \infty$. Then $J_\beta(\lambda)$ is a maximal submodule of $M(\lambda)$ and $L(\lambda,\beta) \cong V(\lambda,\beta)$ for any $\beta \in k^\times$. Moreover, if N is a maximal submodule of $M(\lambda)$, then $N = J(\lambda)$ or $N = J_\beta(\lambda)$ for some $\beta \in k^\times$.*

Proof. (a) It is obvious since $\{x^i \cdot v_\lambda \mid i \geq 0\}$ is a basis of $M(\lambda)$ and each $x^i \cdot v_\lambda$ is a weight vector.

(b) Assume that $M(\lambda) = M_1 \oplus M_2$ for some submodules M_1 and M_2 of $M(\lambda)$. Let $P_1 : M(\lambda) \to M_1$ be the corresponding projection from $M(\lambda)$ to M_1. Then $P_1^2 = P_1$.

If $P_1(v_\lambda) = \sum_{i=0}^{m} \beta_i x^i v_\lambda$, then from $P_1^2(v_\lambda) = P_1(v_\lambda)$, one gets that $P_1(v_\lambda) = \beta_0 v_\lambda$. Thus $M_1 = M(\lambda)$ or $M_1 = 0$. Hence $M(\lambda)$ is indecomposable.

(c) Since $\dim(L(\lambda)) = 1$, $J(\lambda)$ is a maximal submodule of $M(\lambda)$. Obviously, $L(\lambda) \cong V_\lambda$.

(d) If $\lambda = \tau$, then obviously $M(\lambda) \cong M(\tau)$. Conversely, assume that $f : M(\lambda) \to M(\tau)$ is an H-module isomorphism. Then $f(J(\lambda)) = f(x \cdot M(\lambda)) = x \cdot f(M(\lambda)) = x \cdot M(\tau) = J(\tau)$. Hence f induces an H-module isomorphism $\overline{f} : L(\lambda) \to L(\tau)$, $m + J(\lambda) \mapsto f(m) + J(\tau)$, $m \in M(\lambda)$. It follows from (c) and Lemma 4.4(b) that $\lambda = \tau$.

(e) Assume $|\chi| = \infty$. Notice that a submodule of a weight module is still a weight module. Let N be a submodule of $M(\lambda)$. Then $N = \oplus_{\sigma \in \Pi(N)} N_{(\sigma)}$ since $M(\lambda)$ is a weight module. Moreover, $\Pi(N) \subseteq \Pi(M(\lambda)) = \{\chi^i \lambda | i \geq 0\}$. If $\lambda \in \Pi(N)$, then $v_\lambda \in N$ since $0 \neq N_{(\lambda)} \subseteq M(\lambda)_{(\lambda)} = k v_\lambda$, and hence $N = M(\lambda)$ in this case. If $\lambda \notin \Pi(N)$, then $\Pi(N) \subseteq \{\chi^i \lambda | i \geq 1\}$, and hence $N \subseteq \oplus_{i \geq 1} M(\lambda)_{(\chi^i \lambda)} = J(\lambda)$ in this case. It follows that $J(\lambda)$ is the unique maximal submodule of $M(\lambda)$

(f) Assume $|\chi| = s < \infty$. Then one can easily check that $L(\lambda, \beta)$ is isomorphic to $V(\lambda, \beta)$ for any $\beta \in k$. It follows from Theorem 4.5(c) that $J_\beta(\lambda)$ is a maximal submodule of $M(\lambda)$ for any $\beta \in k^\times$.

Since $|\chi| = s < \infty$, $M(\lambda) = \oplus_{0 \leq i \leq s-1} M(\lambda)_{(\chi^i \lambda)}$ and

$$M(\lambda)_{(\chi^i \lambda)} = \mathrm{span}\{x^{ns+i} \cdot v_\lambda \mid n \geq 0\}, \quad 0 \leq i \leq s-1.$$

Let $N \subseteq M(\lambda)$ be a submodule. Then

$$N = \bigoplus_{i=0}^{s-1} N_{(\chi^i \lambda)} \text{ and } N_{(\chi^i \lambda)} = M(\lambda)_{(\chi^i \lambda)} \cap N \supseteq x^i \cdot N_{(\lambda)},\ 0 \leq i \leq s-1.$$

Let $A = k[x^s]$ be the subalgebra of H generated by x^s. Then $M(\lambda)_{(\chi^i \lambda)}$ is a free A-module of rank 1 with A-basis $x^i \cdot v_\lambda$, where $0 \leq i \leq s-1$. Notice that A is a polynomial algebra in one variable x^s over k. Since k is algebraically closed, every maximal ideal of A is of the form $\langle x^s - \beta \rangle$, the principal ideal generated by $x^s - \beta$, for some $\beta \in k$. Hence every maximal A-submodule of $M(\lambda)_{(\chi^i \lambda)}$ has the form $\langle x^s - \beta_i \rangle \cdot (x^i \cdot v_\lambda) = A(x^s - \beta_i) x^i \cdot v_\lambda$ for some $\beta_i \in k$, where $0 \leq i \leq s-1$.

Now assume that N be a maximal H-submodule of $M(\lambda)$. If $N_{(\lambda)} = M(\lambda)_{(\lambda)}$, then $v_\lambda \in M(\lambda)_{(\lambda)} = N_{(\lambda)} \subseteq N$, which implies $N = M(\lambda)$, a contradiction. Hence $N_{(\lambda)} \subsetneq M(\lambda)_{(\lambda)}$, and so $N_{(\lambda)}$ is contained in a maximal A-submodule of $M(\lambda)_{(\lambda)}$.

If $N_{(\lambda)} \subseteq \langle x^s \rangle \cdot v_\lambda$, then $N \subseteq J(\lambda)$ since $N_{(\chi^i \lambda)} \subseteq J(\lambda)$ for all $0 \leq i \leq s-1$, and hence $N = J(\lambda)$ in this case.

Now suppose $N_{(\lambda)} \nsubseteq \langle x^s \rangle \cdot v_\lambda$. Then $N_{(\lambda)} \subseteq \langle x^s - \beta_0 \rangle \cdot v_\lambda$ for some $\beta_0 \in k^\times$. We claim that $N_{(\chi^i \lambda)} \neq M(\lambda)_{(\chi^i \lambda)}$, $1 \leq i \leq s-1$. In fact, if $x^i \cdot v_\lambda \in N_{(\chi^i \lambda)}$ for some $1 \leq i \leq s-1$, then $x^s \cdot v_\lambda = x^{s-i} \cdot (x^i \cdot v_\lambda) \in N_{(\lambda)} \subseteq \langle x^s - \beta_0 \rangle \cdot v_\lambda$, and so $v_\lambda = \beta_0^{-1}(x^s \cdot v_\lambda - (x^s - \beta_0) \cdot v_\lambda) \in \langle x^s - \beta_0 \rangle \cdot v_\lambda$, which is impossible since $\langle x^s - \beta_0 \rangle \cdot v_\lambda \neq M(\lambda)_{(\lambda)}$. This shows the claim. Therefore, $N_{(\chi^{s-1} \lambda)} \subseteq \langle x^s - \beta \rangle x^{s-1} \cdot v_\lambda$ for some $\beta \in k$. Let $0 \leq i \leq s-2$. Then the map $f : M(\lambda)_{(\chi^i \lambda)} \to M(\lambda)_{(\chi^{s-1} \lambda)}$ defined by $f(m) = x^{s-1-i} \cdot m$ is an A-module isomorphism. Moreover, $f(N_{(\chi^i \lambda)}) = x^{s-1-i} \cdot N_{(\chi^i \lambda)} \subseteq N_{(\chi^{s-1} \lambda)}$ and $f(\langle x^s - \beta \rangle x^i \cdot v_\lambda) = \langle x^s - \beta \rangle x^{s-1} \cdot v_\lambda$. Hence $f(N_{(\chi^i \lambda)}) \subseteq f(\langle x^s - \beta \rangle x^i \cdot v_\lambda)$, which implies that $N_{(\chi^i \lambda)} \subseteq \langle x^s - \beta \rangle x^i \cdot v_\lambda$. In particular,

$N_{(\lambda)} \subseteq \langle x^s - \beta\rangle \cdot v_\lambda$, and so $\beta \neq 0$ since $N_{(\lambda)} \nsubseteq \langle x^s\rangle \cdot v_\lambda$. Now we have $N = \sum_{i=0}^{s-1} N_{(\chi^i\lambda)} \subseteq \sum_{i=0}^{s-1}(\langle x^s - \beta\rangle x^i \cdot v_\lambda) = J_\beta(\lambda)$. This forces $N = J_\beta(\lambda)$ since N is a maximal H-submodule of $M(\lambda)$. □

From the proof of Theorem 4.12(e) and (f), one gets the following corollary.

Corollary 4.13. *Let N be an H-submodule of $M(\lambda)$, where $\lambda \in \hat{G}$. If $N \neq M(\lambda)$, then there exists a maximal H-submodule J of $M(\lambda)$ such that $N \subseteq J$.*

Let $k[x]$ denotes the subalgebra of H generated by x

Proposition 4.14. *Let M be an H-module, $\lambda \in \hat{G}$. Then there exists an H-module epimorphism from $M(\lambda)$ to M if and only if there is a weight vector v of weight λ in M such that $M = H \cdot v = k[x] \cdot v$.*

Proof. If there is an H-module epimorphism $f : M(\lambda) \to M$, then $f(v_\lambda)$ is a weight vector of weight λ in M and $M = f(M(\lambda)) = f(H \cdot v_\lambda) = H \cdot f(v_\lambda)$. Conversely, assume there is a weight vector v of weight λ in M such that $M = H \cdot v$. Since $\{x^i \cdot v_\lambda | i \geq 0\}$ is a k-basis of $M(\lambda)$, one can define a k-linear map $f : M(\lambda) \to M$ by $f(x^i \cdot v_\lambda) = x^i \cdot v$, $i \geq 0$. Then it is easy to see that f is an H-module epimorphism from $M(\lambda)$ to M. □

Proposition 4.15. (a) *$\{M(\lambda) \mid \lambda \in \hat{G}\}$ is a set of non-isomorphic indecomposable projective objects in $\mathcal{W}$.*

(b)*If $|\chi| = \infty$, then $\{V_\lambda \mid \lambda \in \hat{G}\}$ is a complete set of simple objects in $\mathcal{W}$.*

(c) *If $|\chi| = s < \infty$, then $\{V_\lambda, V(\sigma,\beta) \mid \lambda \in \hat{G}, [\sigma] \in \hat{G}/\langle\chi\rangle, \beta \in k^\times\}$ is a complete set of simple objects in $\mathcal{W}$.*

Proof. By Theorem 4.12(b), each $M(\lambda)$ is indecomposable. Let $f : M \to L$ be an epimorphism and $g : M(\lambda) \to L$ be a morphism in $\mathcal{W}$. Then $f(M_{(\tau)}) = L_{(\tau)}$ and $g(M(\lambda)_{(\tau)}) \subseteq L_{(\tau)}$ for any $\tau \in \hat{G}$. Hence there exists a weight vector $m \in M_{(\lambda)}$ such that $f(m) = g(v_\lambda)$. Define $\phi : M(\lambda) \to M$ by $\phi(h \cdot v_\lambda) = h \cdot m$, $h \in k[x]$. Since $M(\lambda)$ is a free $k[x]$-module with a $k[x]$-basis v_λ, ϕ is well-defined. Obviously, ϕ is a $k[x]$-module map. Moreover, we have $\phi(g\cdot(x^i\cdot v_\lambda)) = \phi(\chi^i(g)x^i g\cdot v_\lambda) = \chi^i(g)\lambda(g)(x^i\cdot m) = \chi^i(g)x^i\cdot(g\cdot m) = g\cdot(x^i\cdot m) = g\cdot\phi(x^i\cdot v_\lambda)$ for all $g \in G$ and $i \geq 0$. Hence ϕ is an H-module morphism. Furthermore, one can easily check that $f\phi = g$. Hence $M(\lambda)$ is an indecomposable projective object in $\mathcal{W}$ for any $\lambda \in \hat{G}$. Thus, by Theorem 4.12(d), Part (a) follows.

Let V be a simple object in $\mathcal{W}$. Since V is a weight module, one can pick up a nonzero weight vector $v \in V$ with weight λ. Then $V = H \cdot v$ since V is simple. It follows from Proposition 4.14 that V is isomorphic to a quotient module $M(\lambda)/N$ of $M(\lambda)$ modulo a submodule N. Since V is simple, N is a maximal submodule of $M(\lambda)$. Thus, Parts (b) and (c) follow from Theorem 4.12(c), (e), (f), Lemma 4.4(b) and Proposition 4.6(c). □

Now we consider the simple objects and indecomposable projective objects in $\mathcal{W}'$. Note that $H' = H/I$ and $\mathcal{W}' = \mathcal{M}' \bigcap \mathcal{W}$, where I is the Hopf ideal of H as stated in Theorem 3.4.

Corollary 4.16. *Let $\mathcal{S}$ be the complete set of finite dimensional simple H'-modules as given in Corollary 4.10 and Corollary 4.11. Then $\mathcal{S}$ is a complete set of simple objects in $\mathcal{W}'$.*

Proof. It follows from Corollary 4.10, Corollary 4.11 and Proposition 4.15(b) and (c). □

For any $\lambda \in \hat{G}$, let $\overline{M(\lambda)} := M(\lambda)/(I \cdot M(\lambda))$. Then $\overline{M(\lambda)}$ is a weight H'-module. For $m \in M(\lambda)$, let $\overline{m}$ denote the image of m under the canonical projection $\pi : M(\lambda) \to \overline{M(\lambda)}$. We have already known that $I = 0$ if and only if $\chi^{-1}(a)$ is not a primitive n-th root of unity for any $n \geq 2$.

Case 0: $I = 0$. In this case, $\overline{M(\lambda)} \cong M(\lambda)$. Hence $\overline{M(\lambda)}$ is indecomposable by Theorem 4.12(b). Moreover, for $\lambda, \sigma \in \hat{G}$, it follows from Theorem 4.12(d) that $\overline{M(\lambda)} \cong \overline{M(\sigma)}$ if and only if $\lambda = \sigma$.

Now assume that $\chi^{-1}(a)$ is a primitive n-th root of unity for some $n \geq 2$. Then $n \leq |\chi|$ by Proposition 3.5(a).

Case 1: $I = \langle x^n \rangle$. In this case, $\overline{M(\lambda)}$ has a k-basis $\{\overline{v_\lambda}, x \cdot \overline{v_\lambda}, \cdots, x^{n-1} \cdot \overline{v_\lambda}\}$ satisfying the following relations:

$$x \cdot (x^{n-1} \cdot \overline{v_\lambda}) = 0,\ g \cdot (x^i \cdot \overline{v_\lambda}) = \chi^i(g)\lambda(g)x^i \cdot \overline{v_\lambda},\ g \in G,\ 0 \leq i \leq n-1.$$

Hence an argument similar to the proof of Theorem 4.5(a) shows that $\overline{M(\lambda)}$ is indecomposable. Moreover, for $\lambda, \sigma \in \hat{G}$, an argument similar to the proof of Proposition 4.6(b) that $\overline{M(\lambda)} \cong \overline{M(\sigma)}$ if and only if $\lambda = \sigma$. Note that $\overline{M(\lambda)} \cong V(\lambda, 0)$ if $|\chi| = n$.

Case 2: $I = \langle x^n, 1-a^n \rangle$ with $a^n \neq 1$ and $n < |\chi|$. In this case, $(1-a^n)\cdot v_\lambda = (1-\lambda(a)^n)v_\lambda$. If $\lambda(a)^n \neq 1$, $I \cdot M(\lambda) = M(\lambda)$, and so $\overline{M(\lambda)} = 0$. If $\lambda(a)^n = 1$, then it follows from an argument similar to Case 1 that $\overline{M(\lambda)}$ is indecomposable. Moreover, for $\lambda, \sigma \in \hat{G}$ with $\lambda(a)^n = \sigma(a)^n = 1$, $\overline{M(\lambda)} \cong \overline{M(\sigma)}$ if and only if $\lambda = \sigma$.

Case 3: $I = \langle x^n - \beta(1 - a^n) \rangle$ with $\beta \in k^\times$, $a^n \neq 1$ and $n = |\chi|$. In this case, one can easily check that $\overline{M(\lambda)} \cong V(\lambda, \beta(1 - \lambda(a)^n))$. It follows from Theorem 4.5(a) that $\overline{M(\lambda)}$ is indecomposable. Moreover, for $\lambda, \sigma \in \hat{G}$, $\overline{M(\lambda)} \cong \overline{M(\sigma)}$ if and only if either $\lambda = \sigma$, or $[\lambda] = [\sigma]$ and $\lambda(a)^n \neq 1$ $(\sigma(a)^n) \neq 1)$.

Corollary 4.17. *Let $\lambda \in \hat{G}$. If $\overline{M(\lambda)} \neq 0$, then $\overline{M(\lambda)}$ is an indecomposable projective object in $\mathcal{W}'$.*

Proof. Let $\lambda \in \hat{G}$ and assume $\overline{M(\lambda)} \neq 0$. Then $\overline{M(\lambda)}$ is an indecomposable H'-module by the above discussion. Since a quotient module of a weight module is also a weight module, $\overline{M(\lambda)}$ is an indecomposable object in $\mathcal{W}'$. Now let $f : M \to L$ be an epimorphism in $\mathcal{W}'$ $(\subseteq \mathcal{W})$, and $g : \overline{M(\lambda)} \to L$ a morphism in $\mathcal{W}'$ $(\subseteq \mathcal{W})$. Then by Proposition 4.15(a), there exists an H-module map $\phi : M(\lambda) \to M$ such that $f\phi = g\pi$. Since $I \cdot M = 0$, we have $\phi(I \cdot M(\lambda)) = I \cdot \phi(M(\lambda)) = 0$. Hence there is a unique H-module map $\psi : \overline{M(\lambda)} \to M$ such that $\psi\pi = \phi$. Thus, $f\psi\pi = f\phi = g\pi$, and so $f\psi = g$ since π is surjective. Note that ψ is also a morphism in $\mathcal{W}'$. This completes the proof. □

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